Complex Integration & Variables

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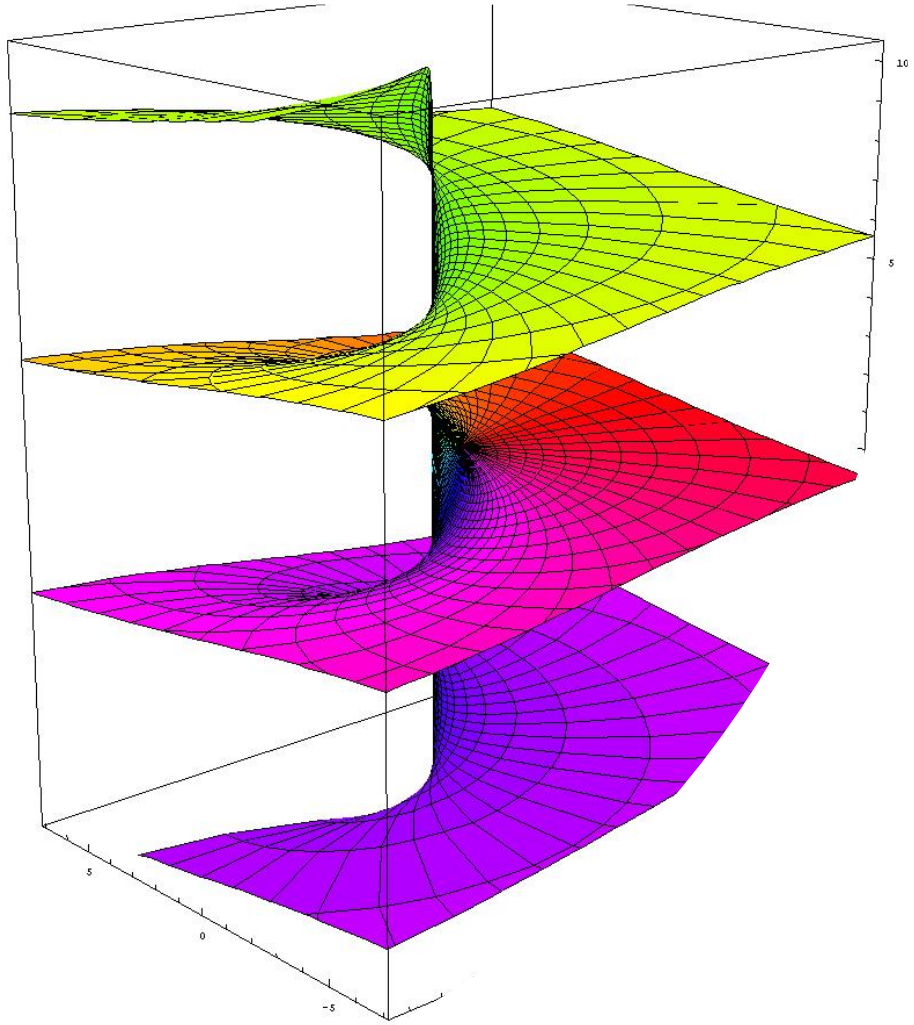
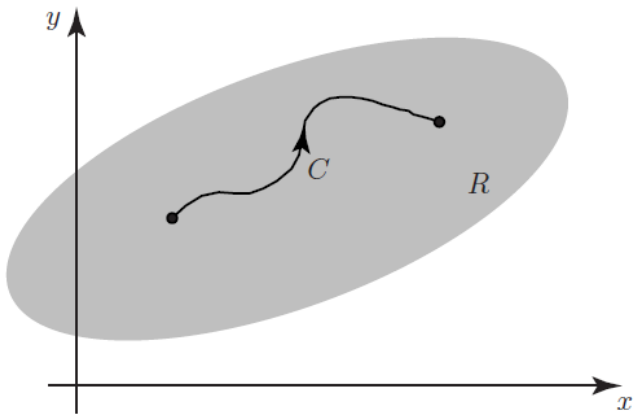


Complex Integration

If $f(z)$ is a single-valued, continuous function in some region R in the complex plane then we define the **integral** of $f(z)$ **along a path** C in R as (see Figure 1)

$$\int_C f(z)\,\mathrm{d}z = \int_C (u + \mathrm{i}v)(\mathrm{d}x + \mathrm{i}\,\mathrm{d}y).$$



Here we have written $f(z)$ and $\mathrm{d}z$ in real and imaginary parts:

$$f(z) = u + \mathrm{i}v \quad \text{and} \quad \mathrm{d}z = \mathrm{d}x + \mathrm{i}\,\mathrm{d}y.$$

Then we can separate the integral into real and imaginary parts as

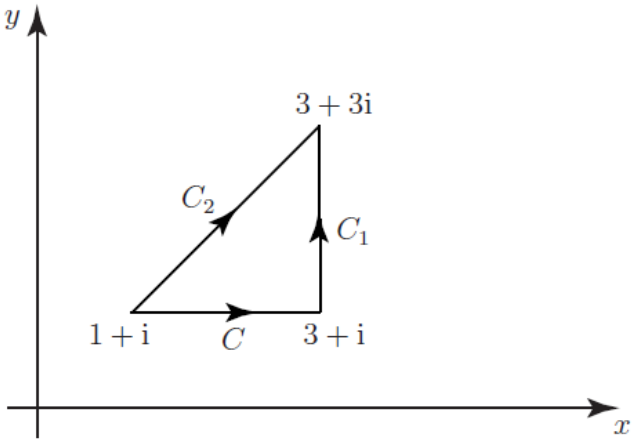
$$\int_C f(z)\mathrm{d}z = \int_C (u\,\mathrm{d}x - v\,\mathrm{d}y) + \mathrm{i}\int_C (v\,\mathrm{d}x + u\,\mathrm{d}y).$$

We often interpret real integrals in terms of area; now we define complex integrals in terms of **line integrals** over paths in the complex plane.

Reference: Mathematical Methods for Physics and Engineering. K.F. Riley ,M.P. Hobson, S.J. Bence, Cambridge University Press

Example Obtain the complex integral: $\int_{C_2} z\,\mathrm{d}z$ where C_2 is the straight line path from

$z = 1 + \mathrm{i}$ to $z = 3 + 3\mathrm{i}$.



We first need to find the equation of the line C_2 in the Argand plane.
We note that both points lie on the line $y = x$ so the complex equation of the straight line is $z = x + \mathrm{i}x$ giving $u = x$ and $v = x$. Also $\mathrm{d}z = \mathrm{d}x + \mathrm{i}\mathrm{d}x = (1 + \mathrm{i})\mathrm{d}x$.

$$\therefore \qquad \int_{C_2} z\,\mathrm{d}z = \int_{C_2} (x\,\mathrm{d}x - x\,\mathrm{d}x) + \mathrm{i}\int_{C_2} (x\,\mathrm{d}x + x\,\mathrm{d}x).$$

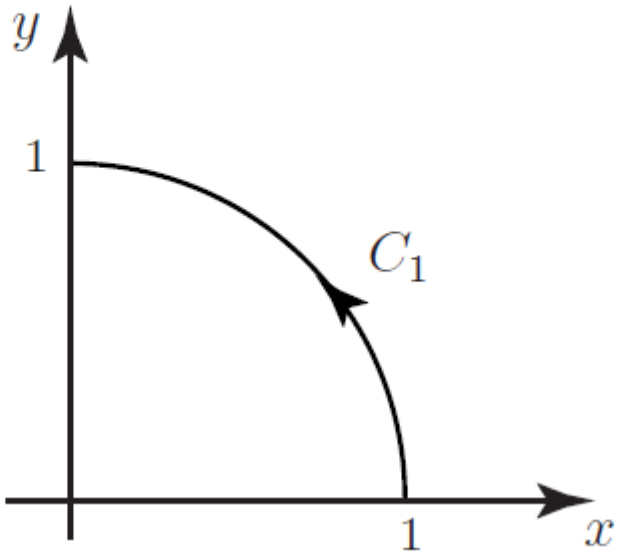
$$= \mathrm{i}\int_{C_2} (2x\,\mathrm{d}x)$$

Next, we see that the limits on x are $x = 1$ to $x = 3$. We are now in a position to evaluate the integral.

Therefore $\int_{C_2} z\,\mathrm{d}z = \mathrm{i}\int_1^3 2x\,\mathrm{d}x = \mathrm{i}\left[x^2\right]_1^3 = i(9 - 1) = 8\mathrm{i}.$

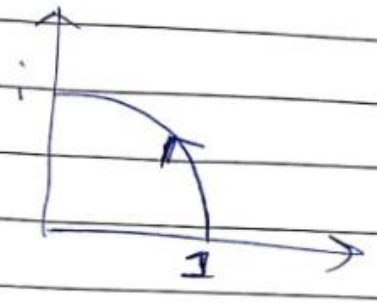
Reference: Mathematical Methods for Physics and Engineering. K.F. Riley ,M.P. Hobson, S.J. Bence, Cambridge University Press

Example As a more intricate example consider the integral $\int_{C_1} z^2 \, \mathrm{d}z$ where C_1 is that part of the unit circle going anticlockwise from the point $z = 1$ to the point $z = \mathrm{i}$. See Figure 3.



Reference: Mathematical Methods for Physics and Engineering. K.F. Riley ,M.P. Hobson, S.J. Bence, Cambridge University Press

$$\int_{C_1} z^2\,dz$$



Substituting $z = e^{i\theta}$

$$\therefore \quad dz = ie^{i\theta}\,d\theta$$

$$\int_{C_1} z^2\,dz = \int_0^{\pi/2} \left(e^{i\theta}\right)^2 ie^{i\theta}\,d\theta$$

θ goes from 0 to $\pi/2$ as C_1 goes from 1 to i

$$= i\int_0^{\pi/2} e^{i3\theta}\,d\theta = \frac{\cancel{i}}{3\cancel{i}}\left[e^{i3\theta}\right]_0^{\pi/2}$$

$$= \frac{\left[e^{i\frac{3\pi}{2}} - 1\right]}{3} = \frac{-[i+1]}{3}$$

An Important Result

$$\boxed{\left|\int_C f(z)\,dz\right| \le \int_c |f(z)||dz| \le M\int_C dl = ML.}$$

M is an **upper bound** on the value of $|f(z)|$ on the path, i.e. $|f(z)| \le M$ on C

L is the length of the path C

What Is an ***Analytic*** Function

A function that is single-valued and differentiable at all points of a domain *R*
is said to be analytic (or regular) in *R*.

A function may be analytic in a domain except at a finite number of points (or an infinite number if the domain is infinite);

Such Finite points are called **Singularities**

Cauchy-Riemann Relations

(to check for analyticity)

If we have a function that is divided into its real and imaginary parts

$$f(z) = u(x, y) + iv(x, y)$$

real

imaginary

to check for analyticity (differentiability)

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.}$$

Necessary and *sufficient* condition is that the four partial derivatives exist, are continuous and satisfy the Cauchy–Riemann relations

►*In which domain(s) of the complex plane is $f(z) = |x| - i|y|$ an analytic function?*

Writing $f = u + iv$ it is clear that both $\partial u/\partial y$ and $\partial v/\partial x$ are zero in all four quadrants and hence that the second Cauchy–Riemann relation in (24.5) is satisfied everywhere.

Turning to the first Cauchy–Riemann relation, in the first quadrant ($x > 0$, $y > 0$) we have $f(z) = x - iy$ so that

$$\frac{\partial u}{\partial x} = 1, \qquad \frac{\partial v}{\partial y} = -1,$$

which clearly violates the first relation in (24.5). Thus $f(z)$ is not analytic in the first quadrant.

Following a similiar argument for the other quadrants, we find

$$\frac{\partial u}{\partial x} = -1 \quad \text{or} \quad +1 \qquad \text{for } x < 0 \text{ and } x > 0, \text{ respectively,}$$

$$\frac{\partial v}{\partial y} = -1 \quad \text{or} \quad +1 \qquad \text{for } y > 0 \text{ and } y < 0, \text{ respectively.}$$

Therefore $\partial u/\partial x$ and $\partial v/\partial y$ are equal, and hence $f(z)$ is analytic only in the second and fourth quadrants. ◄

Reference: Mathematical Methods for Physics and Engineering. K.F. Riley ,M.P. Hobson, S.J. Bence, Cambridge University Press

Power Series and Convergence

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

substituting

$$z = r \exp i\theta,$$

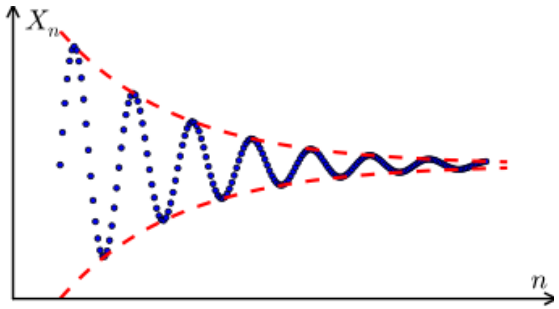
$$f(z) = \sum_{n=0}^{\infty} a_n r^n \exp(in\theta).$$

Converges if:

$$\sum_{n=0}^{\infty} |a_n| r^n$$

Is also convergent

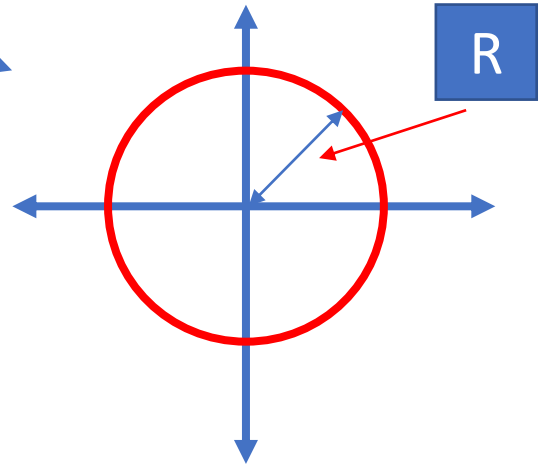
Convergence meaning:
The sum of the series either goes to 0 or a finite value as the terms goes to ∞



Radius and Circle of Convergence

$$\frac{1}{R} = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$$

|z|<R… Convergent
|z|>R… Divergent
|z|=R… Inconclusive



Laurent Series

In mathematics, the Laurent series of a complex function f(z) is a representation of that function as a power series which includes even terms of negative degree. It may be used to express complex functions in cases where a Taylor series expansion cannot be applied.

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-c)^n$$

$$f(z) = \frac{f_0(z)}{(z-z_0)^n} = \frac{a_n}{(z-z_0)^n} + \ldots\ldots\ldots \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + a_1(z-z_0)^2 + \ldots.$$

Nth Coefficient :

$$a_n = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{(z-c)^{n+1}} dz$$

Singularities, Zeros, and Poles

the point $z=\alpha$ is called a singular point, or singularity of the complex function f(z) if f is not analytic at $z=\alpha$ but every neighbourhood of it

Isolated Singularity:

If a function f(z) is not analytic at $z=\alpha$ but analytic everywhere in the entire plane the point $z=\alpha$ is an Isolated Singularity of the function f(z)

$f(z) = \dfrac{1}{1-z}$ has an isolated singularity at $z = 1$

Types of Singularities

Removable | Essential | Pole

Let $f(z)$ have an isolated singularity at α with Laurent series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-\alpha)^n \quad \text{valid for} \quad z \in A(\alpha, 0, R).$$

$$f(z) = \frac{\sin(z)}{z}$$

$$f(z) = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} - \frac{z^{11}}{11!} + \dots\right)$$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \frac{z^8}{9!} - \frac{z^{10}}{11!} + \dots$$

Then we distinguish the following types of singularities at α.

(i) If $c_n = 0$ for $n = -1, -2, -3, \dots$, then we say that $f(z)$ has a removable singularity at α.

(ii) If k is a positive integer such that $c_{-k} \neq 0$ but $c_n = 0$ for $n = -k-1, -k-2, -k-3, \dots$, then we say that $f(z)$ has a pole of order k at α

(iii) If $c_n \neq 0$ for infinitely many negative integers n, then we say that $f(z)$ has an essential singularity at $z = \alpha$.

$$f(z) = e^{1/x} = 1 + \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3}$$

Pole of order 2 at z=1

$$f(z) = \frac{1}{(z-1)^2}$$

Zeros of a Complex Function

if $f(z_0) = 0$ then $z = z_0$ is called a zero of the function $f(z)$

Zeros are classified in a similar way to poles, in that if

$$f(z) = (z - z_0)^n g(z),$$

where n is a positive integer and $g(z_0) \neq 0$, then $z = z_0$ is called a *zero of order* n of $f(z)$. If $n = 1$ then $z = z_0$ is called a *simple zero*. It may further be shown that if $z = z_0$ is a zero of order n of $f(z)$ then it is also a pole of order n of the function $1/f(z)$.

Reference: Mathematical Methods for Physics and Engineering. K.F. Riley ,M.P. Hobson, S.J. Bence, Cambridge University Press

Cauchy's Theorem

Cauchy's theorem states that if $f(z)$ is an analytic function, and $f'(z)$ is continuous at each point within and on a closed contour C, then

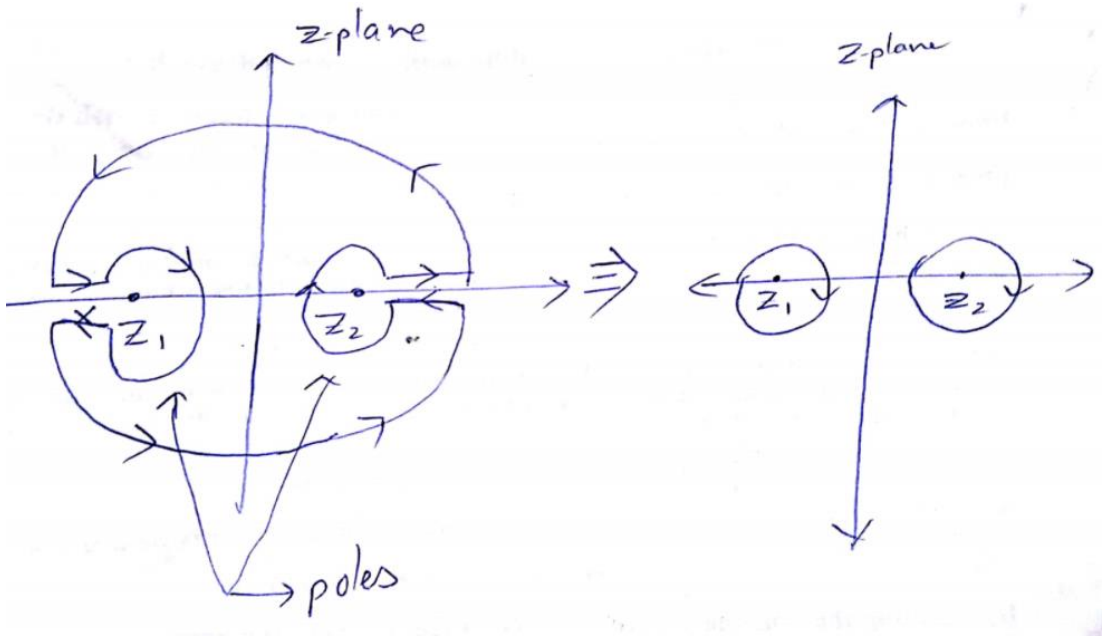
$$\oint_C f(z)\, dz = 0. \tag{24.40}$$

In this statement and from now on we denote an integral around a closed contour by $\oint_C$.

Note: For Proof Refer to Riley Hobson

Most Important

Cauchy's Residue Theorem



If we have a closed contour $\oint_C$ and there are poles inside the contour then we can close the contour around the poles (only poles that are inside the contour)

Select the poles according to the boundary conditions and the specific problem

Take the residue at the selected poles

$$f(z) = \frac{1}{(z+z_1)(z-z_2)}$$

$Z_1, Z_2 > 0$

Calculating Residue's

For a pole of Order 1

Laurent Series

Residue

$$f(z) = \frac{f_0(z)}{(z - z_0)} = \frac{a_{-1}}{(z - z_0)} + a_0 + a_1(z - z_0) + a_1(z - z_0)^2 + \ldots.$$

OR

$$\boxed{Residue|_{Pole=z_0} = \lim_{z \to z_0} (z - z_0) f(z).}$$

Final Answer

$$\oint_C f(z) = 2\pi i \sum_{poles} Residues$$

Calculating Residue's

For a pole of Order 'n'

Laurent Series

Residue

$$f(z) = \frac{f_0(z)}{(z-z_0)^n} = \frac{a_n}{(z-z_0)^n} + \ldots\ldots \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + a_1(z-z_0)^2 + \ldots.$$

OR

$$\text{Residue}|_{Pole=z_0} = \frac{1}{(n-1)!} \lim_{z \to z_0} \left[\frac{d^{n-1}}{dz^{n-1}} (z-z_0)^n f(z) \right]$$

Final Answer

$$\oint_C f(z) = 2\pi i \sum_{poles} \text{Residues}$$

Ex $f(z) = \dfrac{z+3}{z+2}$

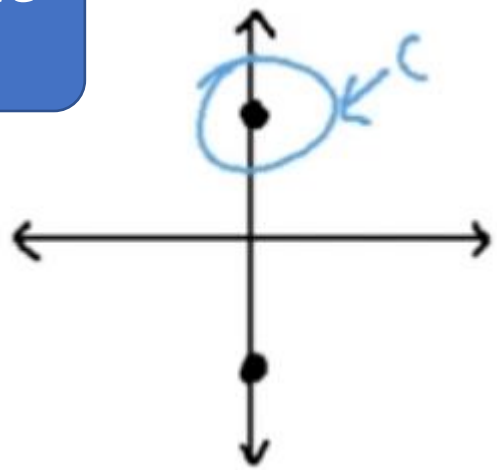
$$\oint_C \frac{z+3}{z+2}\,dz = 2\pi i \operatorname{Res}(f(z), -2)$$

$$\operatorname{Res}(f(z), -2) = \lim_{z \to -2} \frac{\cancel{(z+2)}(z+3)}{\cancel{(z+2)}}$$

$$-2+3 = 1 \checkmark$$

$$2\pi i(1) = \boxed{2\pi i}$$

Taking contour only in the upper half plane



Ex $f(z) = \dfrac{1}{z^2+9} = \dfrac{1}{(z+3i)(z-3i)}$

$$\oint_C f(z)\,dz = 2\pi i \,\mathrm{Res}(f(z), 3i)$$

$$\mathrm{Res}(f(z), 3i) = \lim_{z \to 3i} \frac{\cancel{(z-3i)}(1)}{(z+3i)\cancel{(z-3i)}}$$

$$\frac{1}{3i+3i} = \frac{1}{6i} = \frac{-1}{6} i$$

$$2\pi i\left(\frac{-1}{6} i\right) = \frac{\pi}{3}$$

Ex $f(z) = \dfrac{z+2}{z^4+2z^3-2z-1} = \dfrac{z+2}{(z+1)^3(z-1)}$

$$\oint_C f(z)\,dz = 2\pi i\left(\mathrm{Res}(f(z),1) + \mathrm{Res}(f(z),-1)\right) = 2\pi i\left(\frac{3}{8} - \frac{3}{8}\right) = 0$$

$\mathrm{Res}(f(z),1)$

$$\lim_{z\to 1} \frac{\cancel{(z-1)}(z+2)}{(z+1)^3\cancel{(z-1)}}$$

$$\frac{(1+2)}{(1+1)^3} = \frac{3}{8}$$

$\mathrm{Res}(f(z),-1)$

POLE ORDER 3

$$\frac{1}{(3-1)!}\lim_{z\to -1} \frac{d^2}{dz^2}\frac{\cancel{(z+1)^3}\boxed{(z+2)}}{\cancel{(z+1)^3}\boxed{(z-1)}}$$

$$\frac{1}{2}\lim_{z\to -1}\frac{6}{(z-1)^3} = \frac{6}{(-2)^3} = -\frac{3}{8}$$

Ex $f(z) = \dfrac{e^z}{z^3(z+5)}$ C: Only includes pole at $z=0$

$$\oint_C f(z)\,dz = 2\pi i \operatorname{Res}(f(z), 0) = 2\pi i\left(\frac{1}{2}\cdot\frac{17}{125}\right) = \frac{17\pi}{125} i$$

ORDER = 3

$$\frac{1}{2!}\lim_{z\to 0}\frac{d^2}{dz^2}\frac{\cancel{z^3}e^z}{\cancel{z^3}(z+5)}$$

$$\frac{d^2}{dz^2}\frac{e^z}{z+5} = \frac{(z^2+8z+17)e^z}{(z+5)^3} \Rightarrow \frac{17e^0}{5^3}$$

Ex $f(z) = e^{3/z}$

ESSENTIAL POLE: $z=0$

$$e^z = 1 + \frac{1}{1!}z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \cdots$$

$$e^{3/z} = 1 + \frac{1}{1!}\frac{3}{z} + \frac{1}{2!}\frac{3^2}{z^2} + \frac{1}{3!}\frac{3^3}{z^3} + \cdots$$

$a_{-1} = 3$

$$\oint_C e^{3/z}\,dz = 2\pi i(3) = 6\pi i$$

EX $f(z) = \dfrac{\cos z}{(z-1)^2(z^2+9)}$

POLES: $z = 1, \pm 3i$

$$\oint_C f(z)\,dz = 2\pi i\left(\mathrm{Res}(f(z),3i) + \mathrm{Res}(f(z),-3i) + \mathrm{Res}(f(z),1)\right) = 2\pi i\left(\frac{\cosh 3}{36-48i} + \frac{\cosh 3}{36+48i} + \frac{-\cos 1 - 5\sin 1}{50}\right) = 0.669i$$

$\mathrm{Res}(f(z),3i)$

$$\lim_{z\to 3i} \frac{\cancel{(z-3i)}\cos z}{(z-1)^2(z+3i)\cancel{(z-3i)}}$$

$$\frac{\cos(3i)}{(3i-1)^2(3i+3i)} = \frac{\cosh 3}{36-48i}$$

$\mathrm{Res}(f(z),-3i)$

$$\lim_{z\to -3i} \frac{\cancel{(z+3i)}\cos z}{(z-1)^2\cancel{(z+3i)}(z-3i)}$$

$$\frac{\cos(-3i)}{(-3i-1)^2(-3i-3i)} = \frac{\cosh 3}{36+48i}$$

$\mathrm{Res}(f(z),1)$

ORDER 2

$$\frac{1}{(2-1)!}\cdot\lim_{z\to 1}\frac{d}{dz}\frac{\cancel{(z-1)^2}\cos z}{\cancel{(z-1)^2}(z^2+9)}$$

$$\frac{d}{dz}\frac{\cos z}{z^2+9} = \frac{-2z\cos z - (z^2+9)\sin z}{(z^2+9)^2}$$

$$\frac{-2\cos 1 - (1+9)\sin 1}{(1+9)^2} = \frac{-\cos 1 - 5\sin 1}{50}$$

Proof of Cauchy's Residue theorem by Laurent Series

Laurent Series

$$f(z) = \frac{f_0(z)}{(z-z_0)^n} = \frac{a_n}{(z-z_0)^n} + \ldots\ldots \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + a_1(z-z_0)^2 + \ldots.$$

Nth Coefficient :

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

We are interested in this

So we take n=-1, Thus $(z-z_0)^0 = 1$

$$a_{-1} = \frac{1}{2\pi i}\oint_C f(z)\,dz$$

Thus we are so interested in the Residue: coefficient of the power $(z-z_0)^{-1}$

$$\oint_C f(z)\,dz = (2\pi i)a_{-1}$$

Must See Videos

- https://youtu.be/T647CGsuOVU
- (You think you know about Complex Numbers: Decide after watching the above video: max 2 per day)

Other Videos (For Contour Integration)

https://youtu.be/b5VUnapu-qs

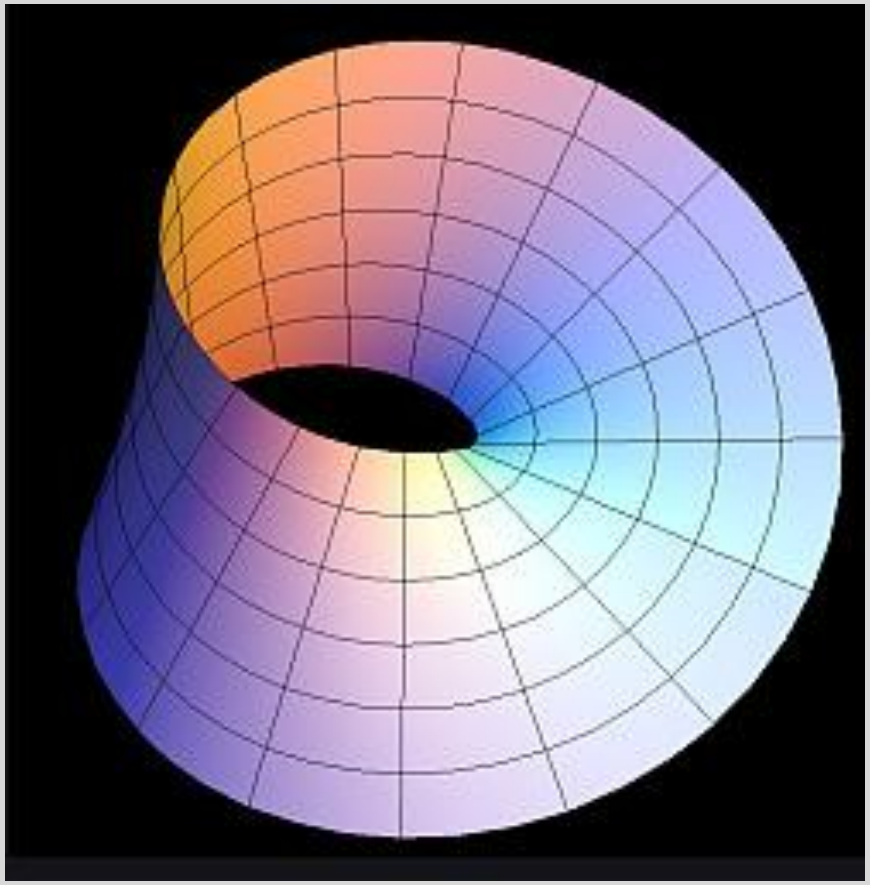
https://youtu.be/_3p_E9jZOU8

References

Mathematical Methods for Physics and Engineering. K.F. Riley ,M.P. Hobson, S.J. Bence, Cambridge University Press

Mathematical Methods for Physicists

A Comprehensive Guide, George B. Arfken, Hans J. Weber and Frank E. Harris, Elsevier



Thank You