The 3-point Problem
as Simultaneous Linear Equations
***** LINEQ Version *****

by Will Frangos

The famous three-point problem is important in structural interpretation. The idea is to determine the strike and dip of a stratum from three points, such as might be encountered in drill holes. The underlying assumption is that the horizon in question is a plane. Graphical methods of solution are commonly employed, lending an intuitive understanding and promoting three-dimensional visualization.

An analytic geometry approach has certain advantages over graphical solution, including precision and rapidity, albeit at the expense of intuitive visualization of the problem. Put succinctly, a mathematical solution would be very handy if you've got 50 of them to do; who wants to perform 50 tedious graphical analyses before dinner?

The method is actually quite straightforward. First, we need to be able to describe an inclined plane by a mathematical function. There are several ways to do it, but let's use a simple slope-intercept form in a standard Cartesian coordinate system;

$$
\begin{aligned}
&z = m_x x + m_y y + z_0 \\
&\text{where } x, y, \text{and } z \text{ are the coordinates,} \\
&\qquad m_x, m_y \text{ are the slopes in the } x \text{ and } y \text{ directions,} \\
&\quad \text{and } z_0 \text{ is the elevation of the plane at the origin} \\
&\qquad\quad \textit{i.e.}, \text{the zero intercept}
\end{aligned}
\qquad \text{Eqn 1}
$$

By this expression we mean that any point on the given plane at lateral location *(x,y)* will have an elevation of z. The tricky bit, of course, is to figure out the values of the slopes and the zero intercept.

Well, if we are given three points on the plane, such as the observed depths in a set of three drill holes of known locations, then we can form a set of simultaneous linear equations and figure out the slopes and zero intercept. Once we know them, we can then predict the elevation at which to expect the horizon at any other location. We can also figure out the strike and dip from the parameters.

The equations look like this:

$$
\begin{aligned}
z_1 &= m_x x_1 + m_y y_1 + z_0 \\
z_2 &= m_x x_2 + m_y y_2 + z_0 \\
z_3 &= m_x x_3 + m_y y_3 + z_0
\end{aligned}
\qquad \text{Eqn 2}
$$

and we want to determine m_x, m_y, and z_0 . Note that x,y, and z are *not* the unknowns here! They are the coordinates in this problem.

We could solve the matter for any set of three points by junior high school methods, but maybe there's a better way using computers. To do that, let's state the set of equations in matrix form, like this:

$$\bar{z} = \bar{\bar{A}}\bar{u}$$

where $\bar{z}$ is a column vector of the observed depths,

$\bar{\bar{A}}$ is the matrix of coefficients, $x_n, y_n,$ and 1 (for the z_0), Eqn 3

and $\bar{u}$ is the vector of unknown parameters to be found, $m_x, m_y,$ and z_0

A parenthetical review of matrix algebra

Eqn 3 above summarizes the set of linear equations, Eqn2, according to the rules for matrix multiplication. The idea is that matrices multiply by summing the products of elements across the rows of the first matrix and down the columns of the second, for example,

$$\begin{matrix} a & b \\ c & d \end{matrix} \quad X \quad \begin{matrix} e & f \\ g & h \end{matrix} \quad = \quad \begin{matrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{matrix}$$

or, with numbers,

$$\begin{matrix} 6 & -1 \\ 0 & 3 \end{matrix} \quad X \quad \begin{matrix} 2 & 1 \\ -3 & 0 \end{matrix} \quad = \quad \begin{matrix} 11 & 6 \\ -9 & 0 \end{matrix}$$

*We note that a square matrix with 1's on the diagonal and 0's elsewhere is called an "identity matrix", **I**, because, when multiplied by another matrix, the result is the same as the latter matrix, i.e., like multiplying by 1. Another important concept is the inverse of a matrix, meaning the matrix which, when multiplied by the original matrix, yields the identity matrix, **I**.*

The solution that we need is then

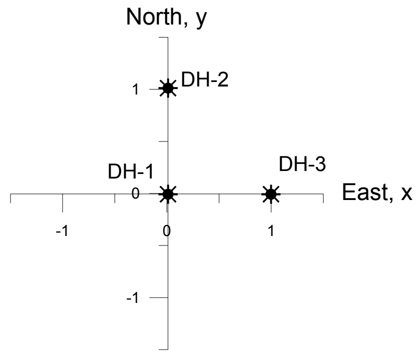
$$\bar{u} = \bar{\bar{A}}^{-1}\bar{z}$$

where $\bar{\bar{A}}^{-1}$ is the inverse of matrix $\bar{\bar{A}}$ Eqn 4

Now, determining the inverse matrix can be an intricate process. Though it can be done by hand, it is best left to a standardized, well-tested computer routine. One alternative is to use the PC program LINEQ, a solver of systems of linear equations. LINEQ doesn't actually calculate the inverse matrix, which, frankly, can be inaccurate and fraught with peril even for modern computers. Instead it uses the LU-decomposition, a more stable techniques.

Example

Let's start with a simple, easily visualized case, as sketched below:



The horizontal coordinates for the drill holes are *(0,0), (0,1),* and *(1,0)*, respectively. Recalling that the coefficient of *z0* is 1 in the general equations, Eqn 2 above, the matrix becomes

$$\overline{\overline{A}} = \begin{matrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{matrix}$$

The simplest case is a plane with no dip at all, *i.e.*, all the depths are the same value. Suppose that depth is 6; then the column vector, $\overline{z}$, is a column of 6's. Let's solve the system and see if we get the right answers.

First, enter the matrix size, 3 in our case, then the nine matrix elements themselves into LINEQ cell by cell. Note that LINEQ starts numbering from 0, so that the first row and first column is called 0,0.

When finished, LINEQ replies by displaying the matrix for your inspection, allowing you to correct any errors. Once you're happy with the entries, go on to enter the depths corresponding to the drillhole locations. LINEQ refers to these as the "right-hand side" of the equation, or RHS. The program solves the system of simultaneous linear equations and displays:

Your answers, Sir/Madam:

X(0)=	0.0000	this is m_x
X(1)=	0.0000	this is m_y
X(2)=	6.0000	and this is z_0

That's what we expected: no slope in the *x*- or *y*-directions, and a zero intercept of 6. Guess we must have got it right.

LINEQ now asks if you want any more sets of depths. You can enter as many sets of these as you like, one after the other.

Exercise: Test the method a little further by specifying a new set of elevations. Try $\bar{z}$ = 6,6,7, corresponding to a plane that dips east with slope of 1. You should get u = 1,0,6. Now try a plane dipping northeast with dips of 1 in each of the north and east directions.

Translation back to Geology:

Being geologists, we'd like to have the plane described as a stratum or horizon with strike and dip, not slope and intercept. We thank the mathematicians for their assistance and proceed as follows:

For the dip, we resolve the two slopes as vectors and convert to angles in degrees. By hand, or using a calculator, we calculate

$$Total\ Slope = \sqrt{m_x^2 + m_y^2}$$

and

$$Dip, \theta = \tan^{-1}(slope)$$

and try to remember to use degrees, not radian, mode.

We can get the dip and strike using the square root and the arctan functions. Depending on your calculator, you might solve the northeast-dipping example done as follows:

atan(sqrt(1*1+1*1))

getting: 54.7356 (if you're in degree mode)

To get the strike direction, we'll use some common sense and the arctan function. We want the direction from north (not east as the mathematicians do it), so we interchange the x- and y-components from what the calculator usually specifies, *i.e.*, we enter atan2(m_x, m_y). For the last exercise above we would enter

atan(1,1) or atan(1/1)

getting: 45

which is the direction of true dip, corresponding to northeast. The strike direction is thus northwest-southeast, so here's where the common sense comes in. We can add or subtract 90°, as appropriate, to express the strike direction.

Check this method to see if you get the right dip and direction for some other easy cases. In particular, set up some beds dipping to the southeast and southwest to illustrate the need for common sense in interpreting the results.