The 3-point Problem
as Simultaneous Linear Equations
*** Excel Spreadsheet Version ***

by Will Frangos

The famous three-point problem is important in structural interpretation. The idea is to determine the strike and dip of a stratum from three points, such as might be encountered in drill holes. The underlying assumption is that the horizon in question is a plane. Graphical methods of solution are commonly employed, lending an intuitive understanding and promoting three-dimensional visualization.

An analytic geometry approach has certain advantages over graphical solution, including precision and rapidity, albeit at the expense of intuitive visualization of the problem. Put succinctly, a mathematical solution would be very handy if you've got 50 of them to do; who wants to perform 50 tedious graphical analyses before dinner?

The method is actually quite straightforward. First, we need to be able to describe an inclined plane by a mathematical function. There are several ways to do it, but let's use a simple slope-intercept form in a standard Cartesian coordinate system;

$$\begin{aligned}
&z = m_x x + m_y y + z_0 \\
&\text{where } x, y, \text{and } z \text{ are the coordinates,} \\
&\qquad m_x, m_y \text{ are the slopes in the } x \text{ and } y \text{ directions,} \\
&\quad \text{and } z_0 \text{ is the elevation of the plane at the origin} \\
&\qquad\quad \textit{i.e.}, \text{the zero intercept}
\end{aligned} \qquad \text{Eqn 1}$$

By this expression we mean that any point on the given plane at lateral location *(x,y)* will have an elevation of z. The tricky bit, of course, is to figure out the values of the slopes and the zero intercept.

Well, if we are given three points on the plane, such as the observed depths in a set of three drill holes of known locations, then we can form a set of simultaneous linear equations and figure out the slopes and zero intercept. Once we know them, we can then predict the elevation at which to expect the horizon at any other location. We can also figure out the strike and dip from the parameters.

The equations look like this:

$$\begin{aligned}
z_1 &= m_x x_1 + m_y y_1 + z_0 \\
z_2 &= m_x x_2 + m_y y_2 + z_0 \\
z_3 &= m_x x_3 + m_y y_3 + z_0
\end{aligned} \qquad \text{Eqn 2}$$

and we want to determine m_x, m_y, and z_0 . Note that x,y, and z are *not* the unknowns here! They are the coordinates in this problem.

We could solve the matter for any set of three points by junior high school methods, but maybe there's a better way using computers. To do that, let's state the set of equations in matrix form, like this:

$$\bar{z} = \bar{\bar{A}}\,\bar{u}$$

where $\bar{z}$ is a column vector of the observed depths, Eqn 3

$\bar{\bar{A}}$ is the matrix of coefficients, $x_n, y_n,$ and 1 (for the z_0),

and $\bar{u}$ is the vector of unknown parameters to be found, $m_x, m_y,$ and z_0

The solution that we need is then

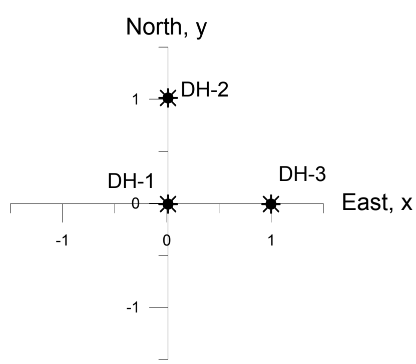
$$\bar{u} = \bar{\bar{A}}^{-1}\,\bar{z}$$

where $\bar{\bar{A}}^{-1}$ is the inverse of matrix $\bar{\bar{A}}$ Eqn 4

Now, determining the inverse matrix can be an intricate process. Though it can be done by hand, it is best left to a standardized, well-tested computer routine. One alternative is to use the common spreadsheet program *EXCEL*; though tedious to set up, EXCEL partially compensates by being widely available.

Examples

Let's start with a simple, easily visualized case, as sketched below:



The horizontal coordinates for the drill holes are *(0,0), (0,1),* and *(1,0)*, respectively. Recalling that the coefficient of *z0* is 1 in the general equations, Eqn 2 above, the matrix becomes

$$\bar{\bar{A}} = \begin{array}{ccc} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{array}$$

The simplest case is a plane with no dip at all, *i.e.*, all the depths are the same value. Suppose that depth is 6; then the column vector, $\bar{z}$, is a column of 6's. Let's solve the system and see if we get the right answers.

First, enter the matrix into a square 3X3 array of Excel's cells, for example, cells B14 through D16. Then calculate the inverse matrix; the Excel formula would be =MINVERSE(B14:D16). In order to get it to work correctly, you need to highlight a square, 3X3 array, enter the formula in a corner cell, and then hold down the Control and Shift keys (Apple and Shift on MacIntosh) as you press Enter.

Excel replies by displaying the inverse matrix in the new array space, for example C21 through E23. It should look like this:

-1	0	1	
-1	1	0	
1	0	0	verifying that we got it right

Now enter the column vector of elevations in another region, for example, F14 through F16:

6
6
6

Now, to solve the system of simultaneous linear equations. Here we will need to multiply the newly calculated inverse by the column vector of depths. The Excel formula for matrix multiplication would be =MMULT(C21:C23, F14:F16). Again, we need to highlight the result array, for example, cells I14 through I16, enter the formula, and hold down Control and Shift as we press Enter.

Excel displays:	0	this is m_x
	0	this is m_y
	6	and this is z_0

That's what we expected: no slope in the *x*- or *y*-directions, and a zero intercept of 6. Guess we must have got it right.

Exercise: Test the method a little further by specifying a new set of elevations. Try $\bar{z}$ = 6,6,7, corresponding to a plane that dips east with slope of 1. You should get u = 1,0,6. Now try a plane dipping northeast with dips of 1 in each of the north and east directions.

Translation back to Geology:

Being geologists, we'd like to have the plane described as a stratum or horizon with strike and dip, not slope and intercept. We thank the mathematicians for their assistance and proceed as follows:

For the dip, we resolve the two slopes as vectors and convert to angles in degrees. By hand, or using a calculator, we calculate

$$Total\ Slope = \sqrt{m_x^2 + m_y^2}$$

and

$$Dip, \theta = \tan^{-1}(slope)$$

and try to remember to use degrees, not radian, mode.

We can get the dip and strike using the square root and the arctan functions. The dip can be done in two Excel cells or one, depending on personal preference. Here is the northeast-dipping example done as one statement (and assuming the same cell references as the example above):

```
=180/pi * atan(sqrt(I14*I14+I15*I15))
```

Excel displays: 54.7356 (the 180/pi converts to degrees)

To get the strike direction, we'll use some common sense and the four-quadrant arctan function. We want the direction from north (not east as the mathematicians do it), so we interchange the x- and y-components from what Excel specifies, *i.e.*, we enter atan2(m_x, m_y). For the last exercise above we would enter

```
=180/pi*atan2(I15,I14)
```

and Excel displays: 45

which is the direction of true dip, corresponding to northeast. The strike direction is thus northwest-southeast, so here's where the common sense comes in. We can add or subtract 90°, as appropriate, to express the strike direction.

Check this method to see if you get the right dip and direction for some other easy cases. In particular, set up some beds dipping to the southeast and southwest to illustrate the need for common sense in interpreting the results.