Thermal Conduction in Permafrost

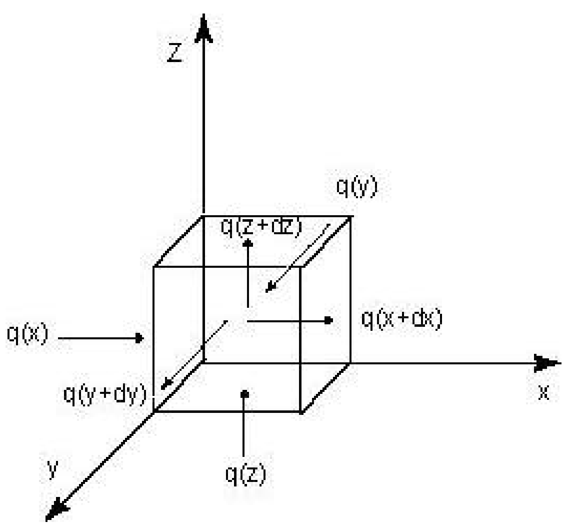
This exercise is based on the work of Lachenbruch, A.H., and Marshall, B.V., 1986, Changing climate: geothermal evidence from permafrost in the Alaskan Arctic, Science, v. 234, p. 689-696. These workers found that temperatures measured in abandoned boreholes drilled for oil exploration reveal a recent warming, probably associated with global warming. They used boreholes in northern Alaska in which mean annual surface temperatures were well below 0 ºC in order to ensure that heat moved in the ground solely by conduction. Under these conditions, flow of heat can be described by Fourier's Law, which states that heat flow across a boundary is equivalent to the thermal conductivity of the material multiplied by the temperature gradient across the boundary:

$Q = -k*(dT/dz)$

The minus sign here is necessary because the flow of heat is in the opposite direction of increasing depth.

In this exercise, students are asked to create a model of heat flow in the outer km of Earth's crust and then to perform some experiments in which air temperature is allowed to change in simulation of Lachenbruch and Marshall's findings.

The equation for temperature at any given depth and at any given time is determined analytically by solving the Thermal Diffusion Equation (see math on next page). This is a second order partial differential equation with a rather involved solution (see the Turcotte and Schubert reading). The derivation of the diffusion equation is given for you on the next page, and is based on the idea that each layer in the model is a box into and out of which heat may flow. The dimensions of the box are dx*dy*dz. This equation is not necessary when casting the problem in STELLA or Fortran (only Fourier's law is necessary), but it useful in explaining how heat travels over time.



$$E(t) = rcTdxdydz$$

$$E(t+dt) = rcTdxdydz + \frac{\partial}{\partial t}[rcTdxdydz]dt + h.o.t.$$

E(t) = energy in box at time t, ρ is density, c is specific heat, T is temperature, E(t+dt) = energy in box at some time dt into the future, h.o.t. = higher order terms of Taylor series expansion

$$\frac{\partial}{\partial t}E(t)dt = \frac{\partial(rcT)}{\partial t}dxdydzdt$$

Change in energy in the box over dt time interval

$$q_x(x) - q_x(x+dx) = -\frac{\partial q_x}{\partial x}dxdydzdt$$

Flux of heat (q) through the box in the x-dimension

$$q_y(y) - q_y(y+dy) = -\frac{\partial q_y}{\partial y}dxdydzdt$$

Flux of heat through the box in the y-dimension

$$q_z(z) - q_z(z+dz) = -\frac{\partial q_z}{\partial z}dxdydzdt$$

Flux of heat through the box in the z-dimension

$$\frac{\partial(rcT)}{\partial t}dxdydzdt = -\frac{\partial q_x}{\partial x}dxdydzdt - \frac{\partial q_y}{\partial y}dxdydzdt - \frac{\partial q_z}{\partial z}dxdydzdt$$

Change in energy in the box over time is equivalent to the sum of the fluxes of heat through the box

so

$$\frac{\partial(rcT)}{\partial t} = -\frac{\partial q_x}{\partial x} - \frac{\partial q_y}{\partial y} - \frac{\partial q_z}{\partial z}$$

Simplify the equation by dividing by dxdydzdt

$$\frac{\partial T}{\partial t} = -\frac{1}{rc}\frac{\partial q_z}{\partial z}$$

Assume 1-dimensional heat flow and get rid of x and y terms

but

$$q_z = -k\frac{\partial T}{\partial z}$$

Fourier's law of heat conduction

so

$$\frac{\partial T}{\partial t} = \frac{k}{rc}\frac{\partial^2 T}{\partial z^2}$$

Substitute in Fourier's law for q

let

$$\mathrm{K} = \frac{k}{rc}$$

Kappa is diffusivity

then

$$\frac{\partial T}{\partial t} = \mathrm{K}\frac{\partial^2 T}{\partial z^2}$$

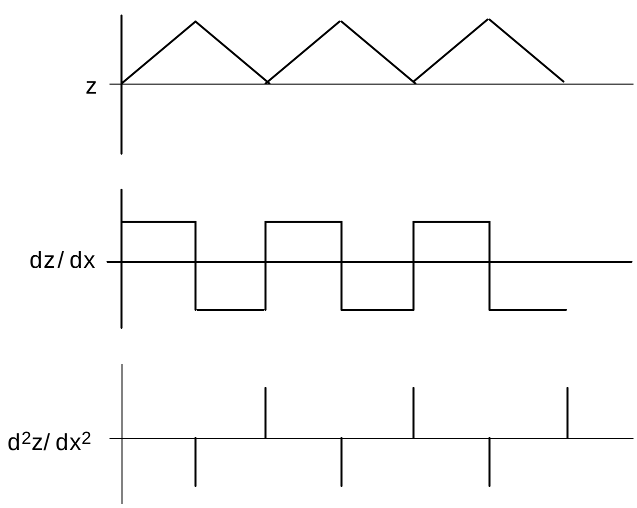
Thermal diffusion equation

The analytical solution to the thermal diffusion equation for any depth and for any time t is:

$$T(t,z) = \overline{T} + \left(\frac{q}{k}\right) * z + T_0 * e^{-\left(\sqrt{\frac{w}{2K}}\right)z} * \cos(wt - \sqrt{\frac{w}{2K}} z)$$

where T-bar is the long term average temperature, T_0 is the amplitude of thermal oscillations, w = (2*π/period of oscillation), and K is the thermal diffusivity (see Turcotte and Schubert reading for more details).

Since the diffusion equation relates the change in temperature with time to the second derivative of the temperature with depth, it is possible to understand the direction of heat flow over time. To see why, consider the following drawing of a hilly topography:

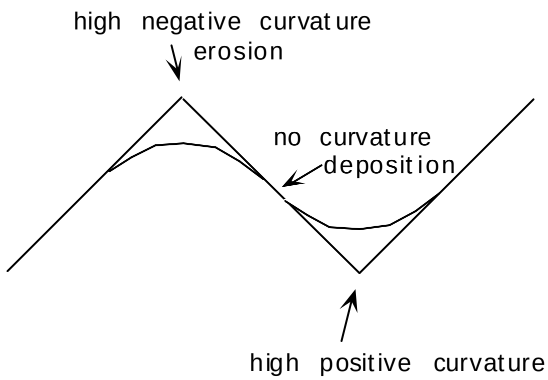


In the figure at the top, we have elevation plotted as a function of distance. In the second figure, the slope is plotted as a function of distance, and in the bottom figure, the curvature (or change in slope with distance) is plotted.

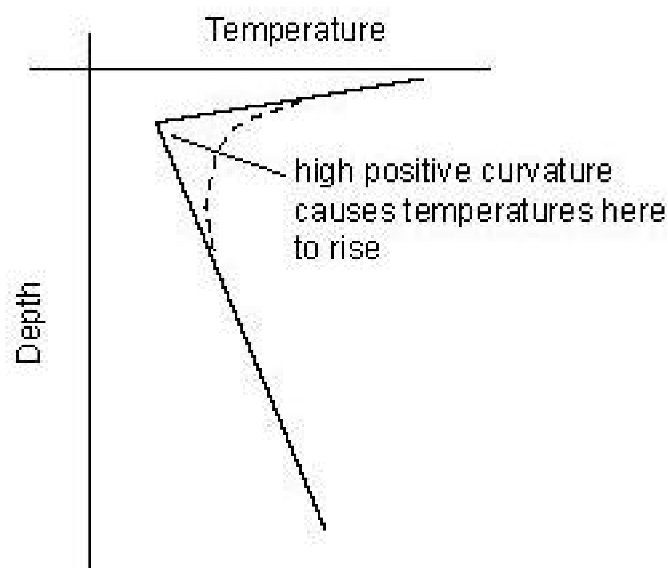
The diffusion equation for topography states that:

$dz/dt = (K)*(d^2z/dx^2)$

or, in other words, the change in elevation with time is related to the curvature of the topography. Where the curvature is negative, erosion occurs. Where it is positive, deposition occurs.



The Thermal Diffusion Equation works in an analogous way. Where there is high negative curvature in the temperature profile, the temperature declines over time, and where there is high positive curvature, the temperature increases. This acts to smooth the profile over time. Consider the following example. After millennia of constant temperature, air temperature suddenly rises, leading to a kink in the thermal profile:



Thermal diffusion will gradually smooth out the profile (see the dashed line) and will eventually cause all of it to shift to the right.

The Heat Flow in Permafrost exercise has several purposes:
1) To teach the students about the basics of heat transfer and the geothermal gradient

2) To introduce oscillatory behavior
3) To introduce the STELLA concept of “bi-flows”
4) To introduce the STELLA "ghost" function in designing and building model structures

In this folder of the blackboard site you will find the following:
1) Copies of the exercise for students in Adobe Acrobat (.pdf) format
2) Copies of the instructor answer key in Adobe Acrobat format
3) A STELLA version of the Heat Flow in Permafrost model
4) A Fortran 90 version of the Heat Flow in Permafrost model
5) An assessment form that can be given to students to determine whether they understood the concepts the exercises are trying to convey