Corks, Buoyancy, and Wave-Particle Orbits

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ABSTRACT

The motion of open ocean waves and associated water-particle orbits constitute a focused topic that occupies a lasting niche in our arsenal of simple, pedagogically compelling things to offer introductory students in the geosciences. The topic provides a vehicle for introducing the general subject of wave behavior, a universally important phenomenon in the geosciences; it challenges students to understand counterintuitive ideas regarding the difference between wave and water-particle motions, serves as a base for advanced topics (for example, oscillatory ripple formation), and is readily "accessible" to students in that wave motions and particle orbits can be easily demonstrated or "tested" in either lab or field conditions. But unless students are exposed to this topic in a manner that goes well beyond introductory-text explanations, possibly in advanced courses, they are not apt to gain an understanding of ocean-wave behavior beyond that provided by purely kinematic explanations. Because of the pedagogical importance of gravity waves – of which ocean waves are an example – and because ambiguities exist in current introductory-text descriptions of waves, we assemble and summarize a dynamical explanation of their behavior. A notable bonus of this explanation is that it provides a simple, concise introduction to the ideas of buoyancy and gravitational forces – also universally important subjects in the geosciences – and how these forces interact during wave motion.

Specifically, the orbit of a water particle beneath a train of ocean waves involves a clockwise motion when viewed from a perspective where the waves are moving from left to right. The vertical component of this motion is associated dynamically with pressure fluctuations about the average pressure state such that the buoyancy force exceeds the gravitational force beneath wave troughs, and the gravitational force exceeds the buoyancy force beneath wave crests. The horizontal component of the motion is associated with these pressure fluctuations wherein the pressure decreases horizontally in the direction of wave motion between wave crests and their leading troughs, and the pressure increases horizontally in the direction of wave motion between wave troughs and their leading crests. These forces alternately accelerate and decelerate the water with simultaneous vertical and horizontal components leading to approximately circular orbits. Peak water (orbit) speeds decrease with increasing wave speed. The attenuation of orbits with increasing depth is related to downward attenuation of the fluctuating pressure field, rather than being attributable to frictional damping.

This dynamical explanation can be concisely presented at several levels. The first involves a qualitative description that appeals to graphical representations of the pressure and velocity fields beneath waves. The second appeals to a simple trigonometric representation of waves and the simplest possible form of Euler's equations combined with the approximation of hydrostatic conditions. The third involves solving linearized forms of Euler's equations.

Keywords: Earth science – teaching and curriculum; education – undergraduate; marine geology and oceanography; miscellaneous and mathematical geology.

INTRODUCTION

For open ocean waves moving left to right, does a water particle near the surface orbit in a clockwise or counterclockwise manner?

If the answer is "clockwise," it is correct. Surprisingly, perhaps, this is not the answer one uniformly gets from geoscientists who are asked this question. In an informal survey of upper-division and graduate geology students, most answered correctly, but a surprising number answered incorrectly. It is our sense that the correct answer typically arises from a visual mnemonic that probably has its origin in introductory-text figures; namely, that the motion at the top of the orbit coincides with that of the wave motion (Figure 1). Also, individuals who have "bobbed" while swimming at the beach, seaward of the breakers, experience this clockwise motion and might justify this answer accordingly. Moreover, our survey of introductory and intermediate-level geology texts and web sites revealed that, whereas the correct sense of orbit motion is presented, in some cases abstracting this information from accompanying figures is not straightforward, and in several cases, misleading explanations are offered. This leads to our next question.

For open ocean waves moving from left to right, *why* does a fluid particle near the water surface orbit in a clockwise manner?

Explanations given by students who have not studied waves typically take the form of a variation on the same visual mnemonic mentioned above; namely (and here we are paraphrasing), that the wave motion "drags" the orbit in a clockwise manner (Figure 1). The introductory texts that we examined essentially sidestep this question. Explanations that are offered normally involve a kinematic, rather than dynamic, description. A notably useful example appeals to envisioning the motion of a cork (or a person) floating on the surface (Figure 2). The vertical component of the orbit is inferred on purely kinematic grounds. Namely, the cork goes up, then the cork goes down with passage of a wave. We are then to gather that, simultaneously, as the leading face of the wave (about halfway between the trough and crest) encounters the cork, the cork begins to move forward with the wave (Figure 2a and 2b). This forward motion occurs as the cork rides up the face of the wave until the wave crest overtakes it (Figure 2b). Then, following passage of the wave crest, the cork does not immediately begin to move down the wave face (thus assuming a backward motion). Rather, we are to infer that the cork retains some of its forward motion – to a point (Figure 2b and 2c). About half-way down the wave face, indeed the cork begins to move down the back face giving it backward motion (Figure 2c and 2d). At the wave trough, some of this backward motion is evidently retained (Figure 2d), as the cork continues to move backward as it starts its upward journey on the next wave face – that is, to a point where it once again begins to move forward (Figure 2e). Here we have come full circle.

This seems reasonable for a cork. But here we must admit that we cannot distinguish whether this sense of reasonableness arises from our having watched corks on waves (and therefore merely accords with what we have observed) or whether

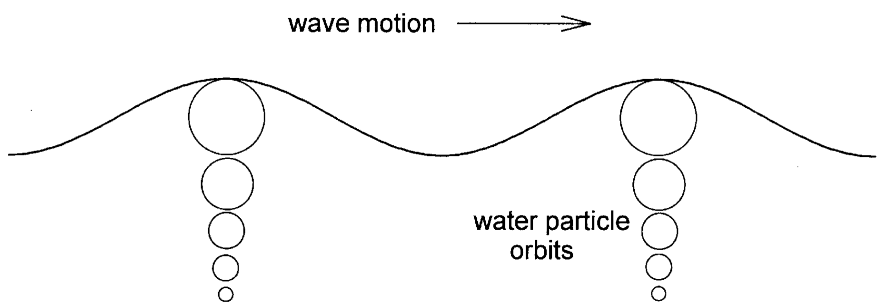

Figure 1. Idealized diagram of water-particle orbits beneath open ocean waves as typically presented in introductory geology text figures; orbit motion is clockwise.

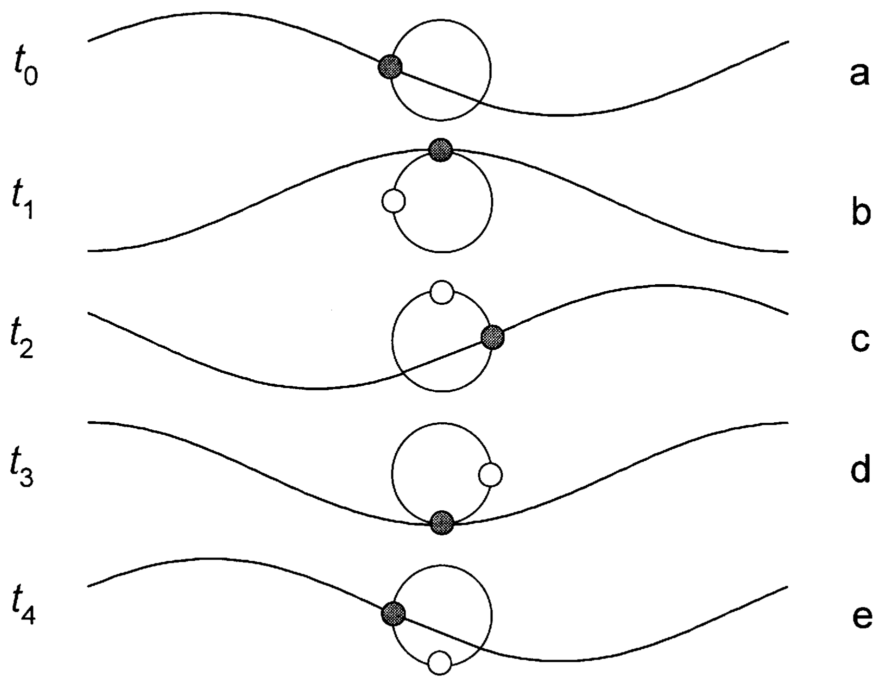

Figure 2. Idealized diagram of orbital motion of floating cork (solid circles) during passage of wave at successive times $t = t_0, t_1, t_2, \ldots$, with previous cork positions indicated (open circles); wave motion is from left to right. Modified from Hamblin (1989, p. 309).

it arises from what we would have deduced about the behavior of a cork had we never observed one. Keep in mind that many have neither observed corks on waves nor attempted to deduce their behavior, such that part or all of the foregoing description may not at all be intuitively appealing. For example, one might reasonably ask why the motion of the cork has to involve any horizontal component at all. Or, why does the cork retain some of its previously attained sense of horizontal motion as it passes through the crest and trough? Or, why does the cork not "slide" down the leading face at the speed of the wave, as does an individual riding a surf board? Or, to include an extreme view, why does wave motion not involve a purely horizontal translation of water mass above a horizontal plane positioned at the level of the wave troughs, or lower, as this (forget the cork!) describes a perfectly reasonable geometrical interpretation of what is occurring (despite defying laws of physics)?

Assuming, momentarily, that we agree that the description of the cork is reasonable and that the cork is mimicking the behavior of the water in its immediate vicinity, we then are normally asked to infer (or simply accept) that this orbiting motion occurs at depth, albeit increasingly attenuated with increasing depth (Figure 1). This, too, seems to us intuitively reasonable, as a water particle beneath the surface is, after all, somehow interconnected with the water around it, including that at the surface, and therefore ought to mimic the orbiting behavior of its neighbors. Moreover, because water, being a fluid, is not rigidly interconnected, this orbiting behavior ought to attenuate with depth inasmuch as information in such a medium is passed spatially with less than perfect efficiency. So, is this explanation satisfactory?

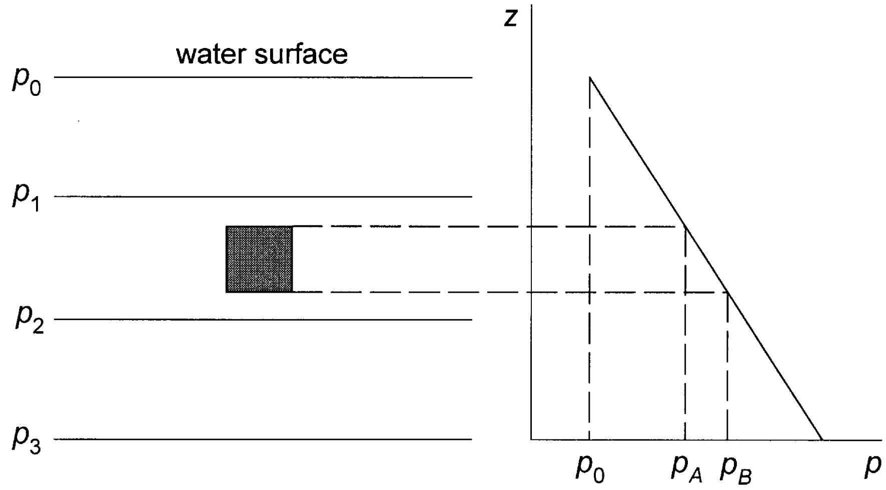

Figure 3. Diagram and plot of static pressure field beneath water surface illustrating uniformly increasing pressure with depth ($p_1 < p_2 < p_3$) above atmospheric pressure (p_0), and different pressures on top (p_A) and bottom (p_B) of static water cube, where $p_A < p_B$.

Our future does not hinge on whether students know the correct sense of motion of particle orbits beneath ocean waves. Nonetheless, judging from its consistent coverage in texts, this topic almost invariably seems to be introduced at some point in the curricula of students in the geosciences. We suspect this is the case for several reasons. First, waves and wave motions are universally important phenomena in the geosciences and, indeed, in all physical sciences. Ocean waves and some of the details of their motion provide a nice example for introducing the general topic of waves. Second, ocean waves provide a nice example of the result – perhaps counterintuitive for some – that mass motion is very different from wave motion. Because counterintuitive ideas can be great fun in classroom discussions, this topic is therefore a pedagogically good topic on which to spend classroom time. Third, a discussion of the oscillatory water motion beneath waves is a basic idea for more advanced topics, for example, near-bed sediment transport and symmetrical ripple formation. Fourth, the oscillatory motion beneath waves is something that students can easily test on their own – with the possible outcome that they may directly gain an appreciation for this special thing that nature invariably does. Moreover, this aspect of wave behavior nicely matches what the texts say ought to happen. That is, it serves as an intuitively compelling example, albeit of modest scope, of a "triumph" of how science unambiguously got its description right.

Focusing on this last item, the exercise of relating water-particle motions beneath a wave to the wave motion is an example of something that is about as close as one can find in nature to an idealized laboratory experiment. There are no surprises: waves and associated water motions are well behaved, and the "experiment" invariably works with virtually no uncertainty. The ultimate reason for this resides in the fact that the waves involve a well behaved (and well understood!) working fluid, as opposed to involving many interacting components as in complex systems – a theme that we more frequently must deal with in the geosciences.

For the reasons outlined above, ocean waves and their particle orbits constitute a focused topic that deserves a lasting niche in our arsenal of simple, pedagogically compelling topics to offer introductory students in the geosciences. But unless this topic is specifically examined in advanced courses, students are apt to see it only at the introductory level, and unless students take a course involving geological fluids, or examine fluid mechanics texts that cover this topic, or have an instructor who treats this topic in a manner that goes well beyond introductory-text explanations, they are not apt to get a dynamical explanation of wave behavior. We therefore decided that it would be useful (and fun!) to assemble and summarize an explanation here, going beyond kinematics, and thereby provide clearer answers to the two questions above. In addition, the explanation provides a simple, concise introduction to the idea of how buoyancy and gravitational forces interact in association with wave motion, and it clarifies a couple of special things about tsunamis. Finally, note that although we focus here on ocean waves, these are only one example of a general class of waves – gravity waves – for which the restoring force is gravitational and which occur in many water environments. This is in contrast to, say, capillary waves, for which the restoring force is surface tension. (The small surface "ripples" that propagate from a fishing line pulled through water are an example of capillary waves.)

QUALITATIVE EXPLANATION OF WAVE-PARTICLE ORBITS

Pressure Fluctuations

Consider a static water column; in this situation, the pressure at a specified position beneath the surface is proportional to the height of the water column above it, such that the water pressure increases linearly with depth (Figure 3). Now envision adding a train of waves to the surface and consider a horizontal line beneath the waves (Figure 4). Applying our rule that pressure increases with depth, the pressure at a position on that part of the line that is beneath a wave crest is, at that instant, momentarily greater than the average pressure at the same level. Similarly, the pressure at a position on that part of the line that is beneath a wave trough is momentarily less than the average pressure at that level.

Does this rule apply to any depth? Consider placing a pressure transducer at various depths. Not too far beneath the waves, the pressure fluctuates about an average value, very nearly equal to the pressure that would occur under static conditions, as waves pass over the transducer. With increasing depth, however, the magnitude of the fluctuations decreases relative to those measured near the surface. This information can be summarized in graphical form (Figure 5), where it is important to note that this represents an instantaneous "snapshot" of lines of equal pressure beneath the waves. One may then envision that this pressure field translates horizontally with the waves in the direction of wave motion. Notice that, relative to average (static) conditions, pressure lines near the surface are closer together beneath wave troughs and farther apart beneath wave crests. This means that, beneath troughs, the rate of pressure increase with depth is greater than it would be under static conditions; and beneath crests, the rate of pressure increase with depth is lower than it would be under static conditions (Figure 5b). We will return to this point momentarily.

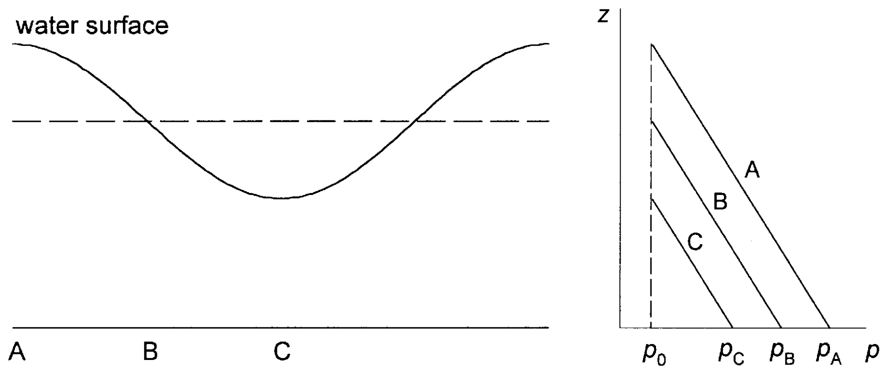

Figure 4. Diagram and plot of instantaneous variations in pressure *p* with vertical position *z* beneath wave crest (A) and trough (C) compared with average conditions (B) that are otherwise associated with average static water surface (dashed horizontal line).

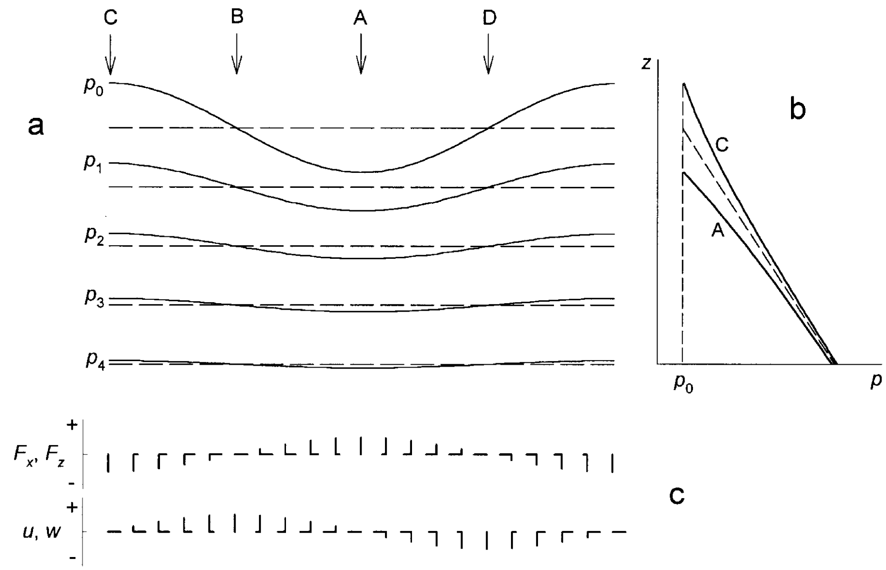

Figure 5. Diagram (a) of fluctuating pressure field beneath waves (solid lines, where $p_0 < p_1 < p_2 < p_3 < p_4$) relative to static pressure conditions (dashed lines); plot (b) of instantaneous variations in pressure *p* beneath trough (A) and crest (C) of wave (sloping solid lines) relative to static conditions (sloping dashed line); and plot (c) of instantaneous net vertical (F_z) and net horizontal (F_x) force components (bold lines) acting on water particles, together with instantaneous vertical (*w*) and horizontal (*u*) velocity components associated with fixed depth; lengths of force and velocity vector components represent relative magnitudes.

Vertical Motions

Consider a small cube of water in the static case (Figure 3). The pressure on the bottom of the cube is greater than the pressure on its top. This suggests that a net upward force (the buoyancy force) acts on the cube. Because the cube is static, however, this buoyancy force must be exactly balanced by a downward force, the weight of the cube.

Now envision the same cube of water positioned within the dynamic pressure field beneath a train of waves (Figure 5). Beneath wave troughs, the difference in pressure between the bottom and top of the cube is greater than this difference under static conditions. This means that the buoyancy force on the cube is greater relative to average conditions, but the weight of the cube remains the same, so a net upward force on the cube exists (Figure 5c). Beneath wave crests, the difference in pressure between the bottom and top of the cube is less than this difference under static conditions. This means that the buoyancy force on the cube is less relative to average conditions, and because the weight of the cube remains the same, a net downward force on the cube exists (Figure 5c).

For reasons that will become apparent, let us "start" the cube at a trough (Figure 5, A). At this position, the net upward force on the cube indeed accelerates it upward. Such a cube reaches its peak upward speed at a position halfway between the trough and crest (B) although, during this interval, the upward buoyancy force is decreasing. At this halfway point, the net vertical force on a cube is zero (buoyancy balances weight), and thereafter, between this point and the wave crest (C), the sign of this net force is reversed. This reversal does not mean, however, that a cube positioned at the halfway point (B) suddenly starts moving downward. Rather, this downward force acts to decelerate the upward motion of such a cube. This deceleration extends to the wave crest, where the vertical speed reaches zero, and the magnitude of the net downward force is at a maximum. Vertical motion then reverses, and a downward acceleration extends to a point halfway between the crest and the next trough (D), where the sign of the net force (buoyancy plus gravitational) again reverses. This provides a deceleration of downward motion extending to the trough (A) where the vertical speed again reaches zero. Thus, the vertical water speed at a fixed position beneath a train of waves tracks the vertical motion of the wave forms passing above it.

Horizontal Motions

Turning to horizontal motions, again consider a horizontal line beneath the waves (Figure 5). Between each wave crest and its leading trough, the pressure along this line decreases in the direction of the trough. Thus, at any instant, a water element positioned on the line within this interval experiences a net force associated with the pressure variation that is directed toward the leading trough (Figure 5c). Conversely, between each wave trough and its leading crest, the pressure along the line increases in the direction of the crest. Thus, at any instant, a water element positioned on the line within this interval experiences a net force directed toward the trough (Figure 5c).

Positioning our water cube halfway between the trough and crest (Figure 5a, B), the horizontal variation in pressure at this position accelerates the cube forward (meanwhile, recall, the cube has attained its peak upward speed here). As the crest overtakes the cube (C), it has accelerated to its maximum forward (horizontal) speed. At this position, the direction of the variation in pressure reverses. This reversal on the back face of the wave begins to decelerate the forward motion of the cube (meanwhile, recall, the cube starts its downward motion). At a position halfway between the crest and trough (D), the horizontal motion has decelerated to zero (recall that this is the position of maximum downward motion). Subsequently, this adverse horizontal pressure gradient accelerates the water cube backward. It reaches its maximum backward (horizontal) motion at the trough (A), where the direction of the horizontal pressure variation once again reverses. Between the trough and a position halfway to the next crest (B), this pressure variation decelerates the backward motion until the horizontal speed reaches zero at the "starting" point. Thus, the horizontal water speed at a fixed position beneath a train of waves oscillates in phase with the wave forms passing above it.

A bit of reflection then leads to the conclusion that these water speeds are consistent with a clockwise particle orbit, where peak vertical speeds are reached at positions halfway between crests and troughs, and peak horizontal speeds are reached at crests and troughs (Figure 5c). An interactive simulation of these orbits has been provided by R.A. Dalrymple at http://www.coastal.udel.edu/faculty/rad/linearplot.html. Our version of this is at http://www.gly.fsu.edu/~parker/waves.html. Both of these simulations, however, are based on linearized versions of the equations governing the water motion and are therefore approximations (see below).

Attenuation of Orbits with Depth

Turning to the attenuation of particle orbits with depth, note that the explanation of orbit motions provided here involves no mention of friction associated with the viscosity of the water. The attenuation of orbits with increasing depth is related to the attenuation of the fluctuating pressure field; it is not attributable to frictional damping (which has been suggested incorrectly in some introductory texts). Specifically, decreasing pressure fluctuations with depth (Figure 5a) translate to decreasing vertical and horizontal pressure variations (about average conditions) that induce water motion.

In turn, the downward attenuation of pressure fluctuations is associated with a downwardly decreasing transfer of momentum from upper fluid layers to lower layers (but not by viscous effects). In effect, each wave possesses a finite amount of potential energy that is locally transformed to kinetic energy of motion, then back again to potential energy, analogous to the harmonic motion of a linear oscillator. Most of this exchange occurs near the surface. That is, because the wave energy is finite, it is completely transformed to finite (and equal) kinetic energy over a short vertical interval.

TRIGONOMETRIC DESCRIPTION OF WAVE MOTION

Some Definitions and Observations

Consider a Cartesian coordinate system where the x-axis is horizontal and positive in the direction of wave motion (Figure 6). The z-axis is vertical and positive upward. Let λ denote the wavelength, a denote the wave amplitude, and h denote the average water depth. Further, let the origin $z = 0$ be positioned at the bed, and $z = \zeta$ denote the local coordinate of the water surface. If c is the wave speed, then the wave period $T = \lambda / c$. For simplicity, assume that the wave is sinusoidal; then for a convenient origin ($x = 0$), the space-time behavior of the wave moving in a direction of positive x can be described by

$$\zeta = h + a\cos(kx - kct) \quad , \qquad (1)$$

where $k = 2\pi/\lambda$ is the wavenumber, $\omega = kc = 2\pi c/\lambda$ is the frequency and t denotes time. This is a solution to the one-dimensional wave equation,

$$\frac{\partial^2 \zeta}{\partial t^2} = \frac{1}{c^2}\frac{\partial^2 \zeta}{\partial x^2} \quad . \qquad (2)$$

To see that (1) is indeed a solution to (2), differentiate (1) twice with respect to time (t) and twice with respect to position x, then substitute these results into (2) to obtain an identity.

Let us make immediate use of (1) in a way that will help clarify particle-orbit motion. Namely, note that for a specified position x, the vertical velocity of a point on the wave form at the water surface is $\partial\zeta/\partial t$ (which, for small-amplitude waves is approximately equal to the vertical velocity component w of a water particle at the surface), so differentiating (1) with respect to time (t) gives

$$\frac{\partial\zeta}{\partial t}=akc\sin(kx-kct) \quad , \qquad (3)$$

which satisfies a special form of the wave equation (2), namely,

$$\frac{\partial\zeta}{\partial t}=-\frac{1}{c}\frac{\partial\zeta}{\partial x} \quad . \qquad (4)$$

This form is only relevant to a wave moving in a direction of positive x.

Although (3) does not fully characterize the vertical component of the water velocity, it nonetheless provides sufficient information to infer that beneath the water surface, vertical motions of water particles track the up and down of the waves passing over them. Moreover, comparing (3) with (1), these vertical speeds are out of phase with the wave form by an amount $\pi/2$.

Help from Euler's Equations

Rather than focusing on details of wave motion, we need only enough information to provide a physical description of particle orbits. In the situation described here, wave dynamics are very well approximated by Euler's equations, the momentum equations for an inviscid flow. This does not imply that seawater possesses zero viscosity. Rather, it merely means that viscous forces are negligible relative to forces associated with pressure and gravity in this problem. Moreover, in the situation where $h \ll \lambda$, a scaling analysis (a form of dimensional analysis; for example, Pedlosky, 1987; Williams and Elder, 1989) of the terms in these equations (APPENDIX) indicates that the horizontal component of the momentum equation reduces to a balance between the local fluid acceleration and the horizontal pressure gradient, and the vertical component of the momentum equation reduces to the hydrostatic approximation. Namely,

$$\frac{\partial u}{\partial t}=-\frac{1}{\rho}\frac{\partial p}{\partial x} \quad , \text{ and} \qquad (5)$$

$$\frac{\partial p}{\partial z}=-\rho g \quad , \qquad (6)$$

where u is the local velocity component parallel to the x-axis, p is pressure, ρ is water density and g is acceleration due to gravity. Equation (5) indicates simply that variations in horizontal motion occur in response to horizontal variations in pressure, and (6) indicates that the pressure varies vertically at a constant rate, as in a static water column.

Integrating (6) with respect to z, then evaluating the result using the boundary condition that $p = p_0$ (atmospheric pressure) at $z = \zeta$ leads to:

$$p=\rho g(\zeta-z)+p_0 \quad . \qquad (7)$$

In turn, differentiating this with respect to x and substituting the result into (5) leads to:

$$\frac{\partial u}{\partial t}=-g\frac{\partial\zeta}{\partial x} \quad . \qquad (8)$$

Thus, in this approximation, the local horizontal acceleration is proportional to the local water-surface slope.

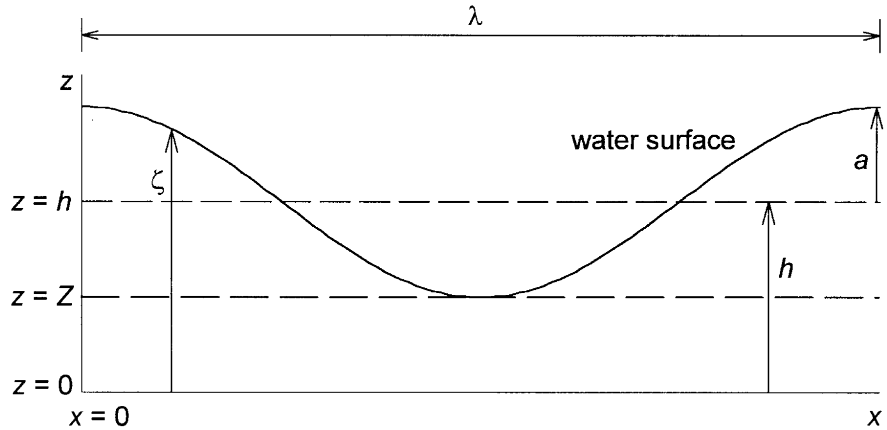

Figure 6. Definition diagram of sinusoidal wave geometry, illustrating wavelength (λ) and amplitude (*a*), average water-surface height (*h*), and height of wave troughs (*Z*).

Differentiating (1) with respect to x gives

$$\frac{\partial\zeta}{\partial x}=-ak\sin(kx-kct) \quad . \qquad (9)$$

Upon substituting this into (8), evidently

$$\frac{\partial u}{\partial t}=gak\sin(kx-kct) \quad . \qquad (10)$$

This is consistent with

$$u=\frac{ga}{c}\cos(kx-kct) \quad , \qquad (11)$$

which is approximately correct only near the water surface, as the development does not fully take into account variations in u with respect to z. Nonetheless, comparing (11) with (1), the local horizontal velocity component (u) is in phase with the wave form.

DYNAMICAL DESCRIPTION OF WAVE MOTION

Linear Inviscid Theory

A slightly more detailed analysis involving linear inviscid theory (APPENDIX) provides the improved, but still approximate, result that

$$u=\frac{ga}{c}e^{-k(Z-z)}\cos(kx-kct) \quad (z\le Z) \quad , \qquad (12)$$

and

$$w=\frac{ga}{c}e^{-k(Z-z)}\sin(kx-kct) \quad (z\le Z) \quad , \qquad (13)$$

where $z = Z$ is the vertical position coinciding with the level of the wave troughs (Figure 6). The exponential parts of (12) and (13) characterize decreasing motion with depth. The peak positive (forward) speed (u_{max}) occurs beneath the wave crest ($z = Z$ and, for example, $x = 0$ and $t = 0$) and is equal to ga/c. Similarly, the peak vertical speed (w_{max}) occurs at a position that is halfway between the crest and trough of the wave.

Orbit Trajectories

Translating (12) and (13) directly to particle orbit speeds is not straightforward. The reason for this is that these expressions are Eulerian in perspective whereas orbit motion is a Lagrangian perspective. That is, (12) and (13) describe the instantaneous velocity at each (x,z) point as viewed by an observer with fixed position. Particle (Lagrangian) orbit speeds, u_p and w_p, in contrast, are those viewed by an observer moving with the particle. Nonetheless, we can provide an approximate connection between these views.

The local fluid (scalar) speed (v) equals

$$\sqrt{u^2+w^2} \quad .$$

For an argument α, $\cos^2\alpha + \sin^2\alpha = 1$, so it follows from (12) and (13) that the local (Eulerian) speed is

$$v = \frac{ga}{c} e^{-k(Z-z)} \quad , \qquad (14)$$

which indicates that this speed varies with vertical position (z). The speed of an orbiting fluid particle in this formulation therefore must vary as it traverses its orbit. Now, an exponential series expansion of (14) gives

$$v = \frac{ga}{c}\left[1 - k(Z-z) + \frac{k^2(Z-z)^2}{2!} - \frac{k^3(Z-z)^3}{3!} + \ldots\right] . \qquad (15)$$

Evidently for small values of $k(Z - z)$, $v \approx ga/c$. For example, with $\lambda = 20$ m and $Z - z = 0.1$ m, the second term within the brackets of (15) is equal to approximately 0.03. That is, this and higher order terms contribute only a few percent to the value of v. A bit of reflection then leads to the conclusion that (12) and (13) are consistent with a clockwise particle orbit where, for small amplitude-to-wavelength waves, particle orbits are approximately circular with, according to (15), approximately constant tangential speed.

Fluid Speed Versus Wave Speed

For shallow-water waves (where $h \ll \lambda$) the (phase) speed (c) equals

$$\sqrt{gh} \quad ,$$

and it follows that $u_{max} < c$ so long as $a < h$, a condition that is required physically. For deepwater waves (where $\lambda \ll h$) the speed (c) equals

$$\sqrt{g/k} \quad ,$$

and it follows that $u_{max} < c$ so long as $a < \lambda/2\pi$, a condition that is required by the linear analysis. Moreover, the appearance of c in the denominator of (12) indicates that the peak horizontal speed (u_{max}) decreases with increasing c. The reason for this is straightforward. Wave motion requires that water be alternately accelerated and decelerated from a speed $u = 0$ to a peak speed $u = u_{max}$ over the wave period $T = \lambda/c$. With increasing wave speed (c), the period during which this can occur decreases. That is, the water is accelerated (or decelerated) for a shorter period and therefore attains a peak speed that is less than when the period "available" for this acceleration is large.

Tsunamis generally are shallow-water waves. Thus, whereas the speed (c) of a tsunami in the open ocean may be very large owing to its large wavelength (λ) and water depth (h), the peak water speed that occurs with passage of a tsunami in the open ocean is very low. For example, water depths in the Pacific Ocean are typically about 4,000 m, so a tsunami travels at a speed of about 200 m s^{-1}. Yet, with an amplitude of one m, the peak water speed (u_{max}) is only about 0.05 m s^{-1}. (Much additional information on tsunamis can be found at http://www.geophys.washington.edu/tsunami/intro.html, including an animation of the propagation of the 1960 Chilean earthquake-generated tsunami across the Pacific, produced by N. Shuto.

CONCLUSIONS

The topic of ocean wave motion and associated water-particle orbits deserves its current niche in our arsenal of pedagogically compelling items to offer introductory students in the geosciences. But unless students are exposed to this topic in a manner that goes well beyond introductory-text explanations, possibly in advanced courses, they are not apt to gain an understanding of ocean wave behavior beyond that provided by purely kinematic explanations. Our summary in this paper of a dynamical explanation of wave behavior includes three levels.

The first involves a qualitative description using graphical representations of the pressure and velocity fields beneath waves and appeals to the intuitive result that pressure fluctuations associated with the passage of waves are attenuated downward. Wave-particle motions are then described as responses to these space-time variations (gradients) in pressure. The second level of explanation appeals to a simple trigonometric representation of waves. When combined with the approximation of hydrostatic conditions – that the pressure increases linearly with depth – this leads to the conclusion that horizontal pressure gradients, which induce horizontal components of particle motion, are locally proportional to the water-surface slope. The third level involves solving linearized forms of Euler's equations, the momentum equations for an inviscid flow. This does not imply that seawater possesses zero viscosity. Rather, it merely means that viscous forces are negligible relative to forces associated with pressure and gravity in this problem. The analysis illustrates how particle orbit speeds vary with depth such that these orbits are only approximately circular for small amplitude-to-wavelength waves.

Several bonuses obtain from a dynamical (versus kinematic) explanation of wave behavior. Namely, the explanation provides a simple, concise introduction to the ideas of buoyancy and gravitational forces and how these interact during wave motion; it illustrates why peak water (orbit) speeds decrease with increasing wave speed; and it illustrates that the attenuation of orbits with increasing depth is related to downward attenuation of the fluctuating pressure field, rather than being attributable to frictional damping.

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REFERENCES CITED

Dalrymple, R.A., 1996, http://www.coastal.udel.edu/faculty/rad/linearplot.html.

Hamblin, W.K., 1989, The Earth's dynamic systems (5th edition): New York, Macmillan, 576 p.

Parker, W.C., 1999, http://www.gly.fsu.edu/~parker/waves.html.

Pedlosky, J., 1987, Geophysical fluid dynamics: New York, Springer-Verlag, 710 p.

Tsunami! (Web site), http://www.geophys.washington.edu/tsunami/intro.html.

Williams, J., and Elder, S.A., 1989, Fluid physics for oceanographers and physicists: Oxford, Pergamon, 300 p.

APPENDIX

The two-dimensional Euler equations for this problem are:

$$u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \quad \text{, and} \qquad \text{(A1)}$$

$$u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \quad , \qquad \text{(A2)}$$

where mass continuity satisfies

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \quad . \qquad \text{(A3)}$$

Each variable quantity in (A1) through (A3) may be written in terms of its time-averaged part, denoted by an upper-case letter, and a small fluctuation about this average, denoted by a prime. That is,

$$u(x,z,t) = U + u'(x,z,t) \quad ,$$
$$w(x,z,t) = W + w'(x,z,t) \quad \text{, and}$$
$$p(x,z,t) = P(z) + p'(x,z,t) \quad . \qquad \text{(A4)}$$

In this problem, $U = 0$ and $W = 0$. Moreover, $\partial P/\partial x = 0$, whereas $\partial P/\partial z = -\rho g$. (Note that U and W are equivalent to space-averaged quantities in this problem, and P is equivalent to a space-averaged quantity if averaging is taken over horizontal planes.) Substitution of the expressions in (A4) into (A1) through (A3) then leads to:

$$u'\frac{\partial u'}{\partial x} + w'\frac{\partial u'}{\partial z} + \frac{\partial u'}{\partial t} = -\frac{1}{\rho}\frac{\partial p'}{\partial x} \quad \text{, and} \qquad \text{(A5)}$$

$$u'\frac{\partial w'}{\partial x} + w'\frac{\partial w'}{\partial z} + \frac{\partial w'}{\partial t} = -\frac{1}{\rho}\frac{\partial p'}{\partial z} \quad . \qquad \text{(A6)}$$

Assuming small fluctuations, and noting that $u = u'$ and $w = w'$, (A5) and (A6) are linearized to:

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p'}{\partial x} \quad \text{, and} \qquad \text{(A7)}$$

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho}\frac{\partial p'}{\partial z} \quad . \qquad \text{(A8)}$$

Differentiating (A3) with respect to time (t),

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial t}\left(\frac{\partial w}{\partial z}\right) = 0 \quad . \qquad \text{(A9)}$$

Because the order of differentiation does not matter, this may be written as

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right) + \frac{\partial}{\partial z}\left(\frac{\partial w}{\partial t}\right) = 0 \quad . \qquad \text{(A10)}$$

Substituting (A7) and (A8) into (A10) then leads to

$$\frac{\partial^2 p'}{\partial x^2} + \frac{\partial^2 p'}{\partial z^2} = 0 \quad . \qquad \text{(A11)}$$

Thus, the fluctuating part of the pressure(p') satisfies Laplace's equation in two dimensions.

Let $z = Z$ denote the position coinciding with the level of the wave troughs. This serves as an upper boundary of an infinite halfspace. For small amplitude waves with $a \ll \lambda$, we may assume that, to a very good approximation, hydrostatic conditions are impressed on this boundary. Then the form of (A11) suggests that

$$p'(x,z,t) = \rho g e^{-k(Z-z)} a\cos(kx - kct) \quad . \qquad \text{(A12)}$$

Differentiating (A12) twice with respect to both x and z confirms that this satisfies (A11). In turn, substitution of the first derivatives of (A12) with respect to x and z into (A7) and (A8) leads to

$$\frac{\partial u}{\partial t} = gke^{-k(Z-z)} a\sin(kz - kct) \quad \text{, and} \qquad \text{(A13)}$$

$$\frac{\partial w}{\partial t} = -gke^{-k(Z-z)} a\cos(kx - kct) \quad . \qquad \text{(A14)}$$

Finally, integrating these with respect to time (t) leads to (12) and (13).