

Computational Geology 9

The Exponential Function

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Introduction

Geology students generally make a close approach to the exponential function in Geology 1 when they learn about radioactive isotopes and geochronology.

The mathematics of the subject is typically presented in a few sentences something like the following. The *half-life* is the time that it takes for half of a given quantity of a given radioactive isotope ("parent") to convert to a radiogenic isotope ("daughter"). For example, if you start with, say, eight million atoms of radioactive isotope *P* (for parent), and the half-life of $P \rightarrow D$ (*D* for daughter) is one thousand years, then, after one thousand years, you will have four million atoms of *P*. Similarly, after two thousand years, you will have two million atoms of *P*; after three thousand years, one million atoms of *P*; and so on. After nine thousand years you will have 15,625 atoms of *P*, or 1/1024 (about 0.1 percent) of the number you started with.

Commonly the succession of fractions (i.e., 1/2, 1/4, 1/8, 1/16, ...) is shown on a graph such as Figure 1. The phenomenon is labeled, appropriately enough, an exponential decay.

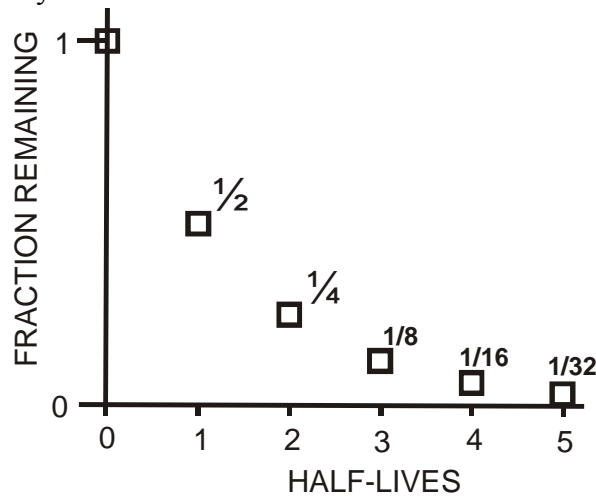


Figure 1. Graph of exponential decay: fraction of parents remaining after successive half-lives.

The equation corresponding to Figure 1, however, is generally not given in Geology 1. Nevertheless, it is straightforward:

$$P = P_0 \left(\frac{1}{2} \right)^{t/t_{1/2}}, \quad (1)$$

where P is the number of parent atoms after an elapsed time, t ; P_0 is the number of parent atoms in the beginning ("time-zero"); and $t_{1/2}$ is the half-life. For example, with $P_0 = 8,000,000$, $t_{1/2} = 1,000$ years, and $t = 9,000$ years, P works out to be 15,625, as you can check with your calculator.

The equation works because the exponent, $t/t_{1/2}$, gives the number of half-lives that have elapsed during the time, t . In Figure 1, only integer values of $t/t_{1/2}$ are plotted, but the equation holds for fractional numbers of half-lives as well.

The 1/2 is in the parentheses, because we are talking about *half*-lives. If we were talking about *third*-lives, the appropriate equation would be

$$P = P_0 \left(\frac{1}{3} \right)^{t/t_{1/3}}, \quad (2)$$

where $t_{1/3}$ is the duration of a third-life. The 1/2 raised to the $t/t_{1/2}$ power gives the *fraction* (P/P_0) of the original parents remaining after t years have elapsed. Clearly, this fraction must be the same as 1/3 raised to the $t/t_{1/3}$ power.

Equation (1) is a gateway to a lot of useful mathematics. In particular, it provides access to the exponential function, the subject of this chapter.

Getting to the Exponential Function.

Let's apply some algebra to Equation 1. Take the natural logarithm of both sides:

$$\ln P = \ln P_0 + \frac{t}{t_{1/2}} \ln(1/2), \quad (3)$$

Recall that the logarithm of a number is minus the logarithm of its reciprocal:

$$\log N = -\log(1/N). \quad (4)$$

Combining Equations 3 and 4 then produces:

$$\ln P = \ln P_0 - \frac{t}{t_{1/2}} \ln(2). \quad (5)$$

Now, both $\ln(2)$ and $t_{1/2}$ are constants. That is, $t_{1/2}$ is constant for the reaction that we are considering; $t_{1/2}$ would have some other value if we were considering some other reaction. Therefore, the ratio of $\ln(2)$ and $t_{1/2}$ must also be a constant (again, specific to this reaction). Call this constant ratio, λ . Thus:

$$\lambda = \frac{\ln 2}{t_{1/2}} . \quad (6)$$

The constant λ is known as the *decay constant*. It has dimensions of time⁻¹, written [T⁻¹].
Now combine Equations 5 and 6:

$$\ln P = \ln P_0 - \lambda t . \quad (7)$$

Taking the antilogarithm of both sides of Equation 7 produces:

$$P = P_0 e^{-\lambda t} . \quad (8)$$

Equation 8 is the usual expression for exponential decay. It is an alternative to Equation 1.

Exponential growth is the opposite of exponential decay. For example, consider a small population of bacteria allowed to multiply without constraint. The number of bacteria after an elapsed time is

$$B = B_0 e^{\beta t} , \quad (9)$$

where B_0 is the initial number of bacteria, and β is the growth constant [T⁻¹]. Such a population doubles after a specific amount of time, t_{2x} :

$$t_{2x} = \frac{\ln 2}{\beta} . \quad (10)$$

The equation analogous to Equation 1 is:

$$B = B_0 2^{t/t_{2x}} . \quad (11)$$

For example, if you start with 100 bacteria, and they double every day, then the number of bacteria would be as follows: after 1 day, 200; after 2 days, 400; after 3 days, 800; after 4 days, 1600; after 7 days, 12,800.

Exponential growth is also known as geometric growth. Recalling Computational Geology 3 (CG-3, "Progressing Geometrically," November 1998), the numbers of bacteria (200, 400, 800, etc.) form a geometric progression. Meanwhile, the numbers of days (1, 2, 3, etc.) form an arithmetic progression. In general, Equations 8 and 9 can be visualized as functions that call for a pairing of geometric progressions against arithmetic progressions. For every arithmetic progression of the independent variable (t , in these examples), there is a corresponding geometric progression of the dependent variable (P or B , here). The *factor* by which the dependent variable increases (or decreases) is determined by the *incremental change* in the arithmetic progression and the rate constant (λ or b , here).

Equations 8 and 9 are variations on the general form,

$$y = e^x , \tag{12}$$

where e is the base of the natural logarithms. Specifically, Equation 8 results from Equation 12 by letting $y = P/P_0$ and $x = -\lambda t$. Similarly, Equation 9 results from Equation 12 by letting $y = B/B_0$ and $x = \beta t$. The function, $y = e^x$ is **the exponential function**. It is unique among functions: it is the only function that, when differentiated, reproduces itself.

Appreciating Euler

Leonhard Euler (1707-1783) was by all accounts the most prolific mathematician of all time. Born in Basel, Switzerland, he completed university there at the age of 15. Mathematically, he was a descendent of Gottfried Wilhelm Leibniz (1646-1716), the Continental inventor of calculus (see CG-5, "If Geology, Then Calculus," March 1999). The lineage went through the famous Bernoulli brothers – James (or Jacob, 1654-1705) and John (Johann, 1667-1548). Both Bernoullis collaborated with Leibniz. Euler's father, a clergyman, studied mathematics under James, and young Euler was tutored by John.

Two of John's sons (Nicholas and Daniel) recommended their friend Euler for appointment at their university, the St. Petersburg Academy of Sciences, which was founded by Peter the Great of Russia at the urging of Leibniz. During political uncertainties in Russia concerning the succession to the throne, Euler decided to end his stay at St. Petersburg (1727-1741). He accepted an invitation of Frederick II of Prussia to join the Berlin Academy of Sciences, which had been founded by Frederick I on the advice of Leibniz. After Berlin (1741-1766), Euler returned to St. Petersburg (1766-1783) at the invitation of Catherine the Great.

By 1771, Euler was almost totally blind. He performed the detailed calculations for his work in his head, and he dictated his papers. His collected works consist of more than seventy volumes.

To set the time frame, this was roughly the time of James Hutton (1726-1797). Euler's life span was very similar to Benjamin Franklin's (1706-1790).

Euler's work established the huge branch of mathematics known as analysis. Simply described, analysis is the mathematics of functions. It includes such curricular subjects as differential and integral calculus, differential equations, complex variables, potential theory, Fourier analysis, and the calculus of variations. Other branches of mathematics are such subjects as number theory, geometry, and algebra.

Euler established analysis in three texts: *Introduction to Analysis of the Infinite* (2 vols., 1748); *Methods of the Differential Calculus* (1755); *Methods of the Integral Calculus* (3 vols., 1768-1770). The three texts were written in Latin. His famous *Introductio* was the precalculus text, and in it he defined the concept of function (the word was first used by Leibniz), the modern concept of logarithm (see CG-3), and the exponential function. His definition of a function was:

A function of variable quantity is an analytical expression composed in any way from this variable quantity and from numbers or constant quantities.

Thus, to Euler, a function was a formula consisting of variables and constants. One of these, which he wrote as lx , he defined as the exponent y such that $a^y = x$. This, of course, is $\log_a x$, the logarithm of x with the base a . In the hands of Euler, the exponential function and e , the base of natural logarithms, followed immediately.

Euler's e and e^x

If you have ever wondered where e came from, this is it. Let x be a finite number, and let ε be an infinitely small number. Then $N = x/\varepsilon$ must be an infinitely large number. Also,

$$x = N\varepsilon \quad (13)$$

We know that $a^0 = 1$. Then, because ε is a number, it must be that

$$a^\varepsilon = 1 + k\varepsilon, \quad (14)$$

where k is also some number. The objective now is to explore this k .

Using the infinitely large N from Equation 13,

$$a^x = a^{N\varepsilon} = (a^\varepsilon)^N. \quad (15)$$

Then, with Equations 13 and 14, Equation 15 becomes:

$$a^x = \left(1 + \frac{kx}{N}\right)^N. \quad (16)$$

The next step is to expand the right side of Equation 16 using the binomial series (which was discovered by Newton in 1665):

$$a^x = 1 + N\left(\frac{kx}{N}\right) + \frac{N(N-1)}{2!}\left(\frac{kx}{N}\right)^2 + \frac{N(N-1)(N-2)}{3!}\left(\frac{kx}{N}\right)^3 + \dots \quad (17)$$

Rewriting Equation 17,

$$a^x = 1 + kx + \frac{1}{2!} \frac{(N-1)}{N} k^2 x^2 + \frac{1}{3!} \frac{(N-1)(N-2)}{N} k^3 x^3 + \dots \quad (18)$$

Now, because N is infinitely large, $N = N-1$, and $N = N-2$, etc., so Equation 18 becomes:

$$a^x = 1 + \frac{kx}{1!} + \frac{k^2 x^2}{2!} + \frac{k^3 x^3}{3!} + \dots \quad (19)$$

This reveals that there is a relationship between the arbitrary base, a , and the number k in Equation 14. Thus, letting $x = 1$, Equation 19 becomes:

$$a = 1 + \frac{k}{1!} + \frac{k^2}{2!} + \frac{k^3}{3!} + \dots \quad (20)$$

In other words, a is a function of k .

To explore these relationships further, one can simply construct Table 1 with the rows corresponding to the terms in Equation 19, and the columns corresponding to various values of kx . The first choice in Table 1 is $kx = 1$. The sum of the first 10 terms is shown. Euler chose to call the infinite sum for this choice e . In other words, Euler called the sum of Equation 20 e for the special case of $k = 1$; i.e., let $a = e$ for $k = 1$ in Equation 20. He calculated the sum to 23 places. Our sum with ten terms gets it correct to 7 significant figures ($e=2.718281828\dots$). The other columns show that different values of k result in different values of a ; hence, a is a function of k .

| | $kx = 1$ | $kx = 2$ | $kx = 2.3026$ |
|---------------|-------------|----------|---------------|
| $(kx)^0/0! =$ | 1 | 1 | 1 |
| $(kx)^1/1! =$ | 1 | 2 | 2.3026 |
| $(kx)^2/2! =$ | 0.5 | 2 | 2.650983 |
| $(kx)^3/3! =$ | 0.16666667 | 1.333333 | 2.034718 |
| $(kx)^4/4! =$ | 0.04166667 | 0.666667 | 1.171285 |
| $(kx)^5/5! =$ | 0.008333333 | 0.266667 | 0.5394 |
| $(kx)^6/6! =$ | 0.001388889 | 0.088889 | 0.207004 |
| $(kx)^7/7! =$ | 0.000198413 | 0.025397 | 0.068092 |
| $(kx)^8/8! =$ | 2.48016E-05 | 0.006349 | 0.019599 |
| $(kx)^9/9! =$ | 2.75573E-06 | 0.001411 | 0.005014 |
| sum= | 2.718281526 | 7.388713 | 9.998697 |

Table 1. First ten terms of Equation 19 for three choices of kx .

After letting $a = e$ for $k = 1$, we can rewrite a number of the preceding equations. Equation 20 becomes:

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots \quad (21)$$

Equation 19 becomes

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \quad (22)$$

For example, with $x = 2$, one gets e^2 , which is the sum in column 2 of Table 1. [You can also note that the $kx = 2$ column produces $e^{2.303}$, or 10; in other words, $\ln(10) = 2.303$.]

In addition, Equation 16 becomes

$$e^x = \left(1 + \frac{x}{N}\right)^N, \quad (23)$$

where, recall, N is an infinitely large number.

In derivations of post-Euler analysis, one can no longer get away with "infinitely large numbers", and Equation 23 is now stated in terms of a limit:

$$e^x = \lim_{N \rightarrow \infty} \left(1 + \frac{x}{N}\right)^N. \quad (24)$$

Letting $x = 1$ in Equation 24, we also have for e :

$$e = \lim_{N \rightarrow \infty} \left(1 + \frac{1}{N}\right)^N. \quad (25)$$

All this, as you have seen, follows *algebraically* from the definition of logarithm. Euler's *Introductio* was a precalculus text.

But the extraordinary nature of e^x is revealed by calculus. To see it, differentiate Equation 22:

$$\frac{d}{dx}(e^x) = \frac{d}{dx} \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right).$$

Completing the term-by-term differentiation produces:

$$\frac{d}{dx}(e^x) = 0 + 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \quad (26)$$

In other words, from Equations 22 and 26,

$$\frac{d}{dx}(e^x) = e^x. \quad (27)$$

The derivative of the exponential function is the exponential function. The consequences of this fact are enormous.

Going Beneath the Exponential Function

To see the importance of Equation 27, consider the following function:

$$y = Ae^{ax}. \quad (28)$$

From Equation 22, this function can be expanded as:

$$y = A \left(1 + \frac{\alpha x}{1!} + \frac{\alpha^2 x^2}{2!} + \frac{\alpha^3 x^3}{3!} + \frac{\alpha^4 x^4}{4!} \dots \right). \quad (29)$$

Differentiating Equation 29,

$$\frac{dy}{dx} = A \left(0 + \frac{\alpha}{1!} + \frac{2\alpha^2 x}{2!} + \frac{3\alpha^3 x^2}{3!} + \frac{4\alpha^4 x^3}{4!} + \dots \right),$$

which simplifies to

$$\frac{dy}{dx} = \alpha A \left(1 + \frac{\alpha x}{1!} + \frac{\alpha^2 x^2}{2!} + \frac{\alpha^3 x^3}{3!} + \dots \right).$$

Now, recognize the function in parentheses as $e^{\alpha x}$, and so

$$\frac{dy}{dx} = \alpha A e^{\alpha x}. \quad (30)$$

Finally, combine Equations 28 and 30:

$$\frac{dy}{dx} = \alpha y, \quad (31)$$

which is the very important result. You may appreciate it more, if it is rewritten with the independent variable as t , for time:

$$\frac{dy}{dt} = \alpha y. \quad (32)$$

In other words, the function

$$y = A e^{\alpha t} \quad (33)$$

has the property *that its rate of change is directly proportional to the amount of y that is present*. The function given in Equation 33 is geometric growth (recall Equation 9). The "rate equation" (Equation 32) for geometric growth is simply that the growth rate is proportional to amount.

Similarly, the function

$$y = A e^{-\alpha t}, \quad (34)$$

which is the equation of exponential decay (recall Equation 8), is underlain by

$$\frac{dy}{dt} = -\alpha y . \quad (35)$$

Equation 35 is the rate equation for exponential decay. If something diminishes with a constant half-life, its rate of decay is proportional to the amount that is present.

Speaking of Radioactive Decay

Is this a reasonable result? I mean, is it reasonable that a radioactive isotope, which has a constant half-life, would decay according to Equation 35? Let's take an example.

The half-life of the reaction $^{14}\text{C} \rightarrow ^{14}\text{N}$ (the decay of radiocarbon) is 5,730 yr, as you probably know. From Equation 6, the decay constant (λ) for this reaction is:

$$\lambda = \frac{\ln 2}{5,730} = 1.21 \times 10^{-4} \text{ yr}^{-1} . \quad (36)$$

From Equation 8, the number of ^{14}C atoms remaining in, say, a sample of plant carbon as a function of time is then:

$$P = P_0 e^{-(0.000121)t} , \quad (37)$$

where the number P_0 is set at the time the plants die. This means that the number of ^{14}C atoms present in the sample decreases by the rate reaction:

$$\frac{dP}{dt} = -(1.21 \times 10^{-4})P . \quad (38)$$

Equation 38 can be rearranged to produce:

$$\frac{dP}{P} = -(1.21 \times 10^{-4})dt . \quad (39)$$

The left side of Equation 39 has an important meaning. The numerator (dP) is the number of ^{14}C atoms in the sample that become ^{14}N atoms in the interval of time dt . The denominator (P) is the number of ^{14}C atoms present during that time interval (which is tiny enough that we don't worry about the fact that P changes during the time interval). The ratio dP/P is the *probability* that any one of the ^{14}C atoms becomes a ^{14}N atom during the time interval. Equation 39, therefore, says that the probability that a ^{14}C atom becomes a ^{14}N atom during any given year is 1.21×10^{-4} , or 0.012 percent.

As you probably know from Geology 1, the conversion of a ^{14}C atom to a ^{14}N atom occurs when a neutron in the ^{14}C nucleus throws out a β^- particle. In such an event, the neutron becomes a proton, the atomic number increases (hence the carbon atom becomes a nitrogen atom), and the mass number is unchanged (at 14). This is called β^- -decay.

^{87}Rb is another atom with an unstable nucleus in which a neutron emits a β^- particle. The decay constant for the reaction $^{87}\text{Rb} \rightarrow ^{87}\text{Sr}$ is $1.42 \times 10^{-11} \text{ yr}^{-1}$, which means that the probability is 1.42×10^{-9} percent that any given ^{87}Rb atom would emit a β^- particle in any given year. Clearly, it is much more unlikely that a β^- particle is ejected from a ^{87}Rb nucleus than from a ^{14}C nucleus. As a result, the half-life of a collection of ^{87}Rb atoms is much longer than the half-life of a collection of ^{14}C atoms. From Equation 6 and the λ for ^{87}Rb decay, the half-life is 48.8 billion years.

It is important to recognize that the probability dP/P pertains to a huge quantity of P atoms. It is these huge numbers that assure the reliability of the mathematical rate law underpinning Equations 1 and 8. Here is how G.B. Dalrymple explains it in *The Age of the Earth* (Stanford University Press, 1991, p. 83-84):

Radioactive decay is most easily understood as a statistical process in which each atom of a given radioactive nuclide has exactly the same probability of decaying in some particular period of time as any other atom of that nuclide....

Because each atom decays independently of all others, the more atoms there are, the more decays occur, and so the number of atoms that decay per unit time is directly proportional to the number of atoms of that species present....

Because decay is a statistical process, it is not possible to tell when any particular atom will decay. For a small number of atoms, therefore, it is virtually impossible to determine the exact number of decays in a given time....

Fortunately, the numbers of atoms in even small amounts of matter are very large – as little as 0.00001 gram of potassium contains 150,000 trillion atoms! – so the statistical nature of radioactive decay is of no practical concern to the accuracy of radiometric dating.

Clearly, the eight million radioactive parent atoms mentioned at the beginning of this article are only to illustrate the mathematics. The number needs to be many orders of magnitude larger to represent the actual application. The same is true of the numbers of bacteria mentioned in connection with Equation 9.

As noted in the above excerpt, the decay of one atom is independent of the decay of the others around it. In other words, the probability that a given atom ejects a β^- particle (or decays by one of the other decay processes) in a given amount of time is not affected by the number of surrounding unstable atoms. If this were not so, then the decay constant and half-life of the reaction would vary with P . For example, if the probability of an atom's decay were increased by the presence of nearby unstable atoms, then the probability of the atom's decay would decrease with time as the pile of unstable atoms converted to stable ones. Accordingly, the decay constant would decrease with time, and half-life would increase with time. In such a case, the mathematics of constant half-lives would result in a confusion of different ages calculated from different isotopes.

That the half-life is independent of time (or extent of reaction) is only one way that the rate constant and half-life are constant. The mathematics of constant half-lives also produces consistent results on rocks that have experienced diverse histories. Thus the rate constants are not affected by changing physical or chemical conditions. Temperature, pressure, and chemical reactions do not affect the probability that a given atom decays in a given amount of time. It is in this sense, also, that the rate constant and half-life are a constant for any given reaction.

If the probability that a P atom becomes a D atom were affected by the amount of P atoms present, the amount of other atoms present, the temperature, pressure, or chemical reactions, then radiometric dating would be a mess. The mathematics, too, would be unpleasant. The decay rate would not be proportional to the amount of parent isotopes present; the rate constants would not be constant; the half-lives would not be constant; Equations 1 and 8 would not hold.

Going the Other Way

Equation 38 shows the derivative dP/dt on the left side of the equation and the function $P(t)$ on the right side of the equation. In Equation 39, the derivative has been broken up: the dP part of it remains on the left, and the dt part of it has been moved to the right. This is an important step.

As discussed in CG-5, the notation dy/dx dates to Leibniz's calculus and has been called the differential coefficient. Leibniz thought of dy/dx as the ratio of two very tiny quantities: dx as a tiny change in the independent variable, and dy as the change in the dependent variable brought about by the tiny change in the independent variable. By this view of the derivative as a ratio, the two parts can be separated. As it turns out, this is the critical first step in solving many rate problems. A rate problem is one in which you have an equation like Equation 35 and you want to find the function, y , that satisfies that equation. In other words, given Equation 35, how do you find Equation 34?

Starting with Equation 35, the first step is to "break up the derivative" and move the dt to the other side, and move the y to the left:

$$\frac{dy}{y} = -\alpha dt . \quad (40)$$

This step, appropriately enough, is known as separating variables. Next integrate both sides:

$$\int \frac{1}{y} dy = -\int \alpha dt . \quad (41)$$

Integrating the left side produces:

$$\int \frac{1}{y} dy = \ln y + c_L , \quad (42)$$

where c_L is the constant of integration obtained from integrating the left side of Equation 41. Integrating the right side of Equation 42 produces:

$$-\int \alpha dt = -\alpha t + c_R , \quad (43)$$

where c_R is the constant of integration. Then combining Equations 41, 42, and 43 produces:

$$\ln y = -\alpha t + c , \quad (44)$$

where c is a constant, namely $c_R - c_L$.

Now the problem is to find c . This requires additional information than that contained in Equation 35. You can find c , if you know the value of y at some time, t . Let's say that the time for which you know y is time zero (i.e., $t = 0$), and that the value of y then is A ; that is,

$$y|_{t=0} = A . \quad (45)$$

(The vertical line is read "at"; therefore, Equation 45 is read "y at $t = 0$ equals A .") To find c , simply combine Equations 44 and 45; that is, let $t = 0$ and $y = A$ in Equation 44:

$$\ln A = 0 + c . \quad (46)$$

From this we clearly have found c :

$$c = \ln A . \quad (47)$$

To complete the solution, combine Equations 44 and 47:

$$\ln y = -\alpha t + \ln A . \quad (48)$$

Rearrange:

$$\ln \frac{y}{A} = -\alpha t . \quad (49)$$

Exponentiate both sides:

$$\frac{y}{A} = e^{-\alpha t} . \quad (50)$$

Simplify:

$$y = Ae^{-\alpha t} . \quad (51)$$

This result is the same as Equation 34. Thus, we have gone from the rate equation (Equation 35) back to the function (Equation 36) that is underlain by that rate function.

Arriving at Differential Equations

The process that we have just worked through is one of the most important in using calculus to solve problems. The process starts with an equation that contains one or more derivatives. Such an equation is called a *differential equation*. As illustrated by this example, the solution of a differential equation is a function (Equation 51). This makes solving differential equations something different than solving algebraic equations (such as $x^2 + 5x + 6 = 0$). When you solve an algebraic equation, you find one or more

numbers that satisfy the equation. When you solve a differential equation, you find a *function* that satisfies the equation.

Part way through our example (solving Equation 35), we needed to bring in additional information to get the final solution (Equation 51). The additional information was the value of the function at a specified value of the independent variable (Equation 45). This additional information is known as a *boundary condition*. Because solving differential equations involves integration, and integration produces constants of integration, it is impossible to work out the complete solution without boundary conditions. If your differential equation involves a function of a single variable (called an *ordinary* differential equation, which is the case for Equation 35), you need one boundary condition for each time that you integrate. If, for example, your differential equation contains the second derivative of the unknown function, you will need to integrate twice, and so you will need two boundary conditions; such equations are called second-order differential equations. The example of Equation 35 is a first-order differential equation, requiring only one boundary condition. Because the independent variable is time, and the boundary condition was set at $t = 0$, it is called an *initial condition*.

Concluding Comments

Geology is about rates and time. A differential equation is an equation expressing rates. Solving it produces a function that describes how the quantity in question varies with time. Solving differential equations, therefore, allows one to explore the consequences of making various assumptions about rates. Clearly such deduction – a form of modeling – is of interest to geologists.

One of the simplest assumptions, or models, that can be made is that the rate of change of a quantity is directly proportional to the amount of the quantity. For reasons explained by Dalrymple in the above excerpt, there are good *a priori* reasons to think that that simple statement about rates would apply to the phenomenon of radioactive decay. The consequence of that rate law's applying to radioactive decay is that the quantity of parents as a function of time is described by a simple exponential function.

For many other phenomena, other statements about rates are reasonable. These produce other functions. In many cases, these more-complicated functions *contain* the exponential function as a part. When one goes from rates to functions, one encounters the exponential function repeatedly. There is no avoiding Euler's e or his e^x .

Sources

The sources for the historical information on Leonhard Euler were *A History of Mathematics, An Introduction* by Victor J. Katz (2nd ed., Addison-Wesley, Reading MA, 1998) and *The Historical Development of the Calculus* by C.H. Edwards, Jr. (Springer-Verlag, New York, 1979). Chapter 10, "The Age of Euler", in *The Historical Development* is particularly instructive. The derivation of Euler's e and e^x here closely follows the section entitled "Euler's exponential and logarithmic functions" (p. 272-274).