

Computational Geology 8

The Power Function

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Introduction

In the last “Computational Geology” (CG-7, “The Algebra of Unit Conversions,” September 1999), the Hack equation (Hack, 1957) came up as an example of an empirical equation for which one needs to be particularly careful with units. The equation,

$$L = 1.4A^{0.6}, \quad (1)$$

relates stream length (L , mi) to drainage-basin area (A , mi²).

In CG-6 (“Solving Problems,” May 1999), Stokes' Law was used to find the travel time for a dead foraminifer to settle to the abyssal plain. This equation, which relates settling velocity (v) of a small, spherical particle to the densities of the settling particle and fluid (ρ_s and ρ_f , respectively), the viscosity of the fluid (μ), and the diameter of the particle (d), is

$$v = \frac{\Delta\rho g}{18\mu} d^2, \quad (2)$$

where $\Delta\rho$ is the difference in densities, $\rho_s - \rho_f$. Stokes derived the equation from consideration of the driving and resisting forces. Any set of consistent units can be used for the variables. (The 1/18 assumes that the particle is a sphere).

The empirically inferred Hack equation and the *a priori* Stokes' Law are both examples of power functions, the general form for which is

$$y = ax^b. \quad (3)$$

For Hack's equation, $y=L$, $x=A$, $a = 1.4 \text{ mi}^{-0.2}$, and $b=0.6$. For Stokes' Law, $y = v$, $x = D$, $a = \Delta\rho/18\mu$, and $b = 2$. This column is an appreciation of Equation 3.

About $y = ax^b$

The signal fact about Equation 3 is what happens when one takes logs. The equation is transformed to

$$\log y = \log a + b \log x, \quad (4)$$

which is the equation of a straight line of $\log y$ vs. $\log x$. If $\log y$ is plotted against $\log x$ on normal graph paper (Fig. 1a), or if y is plotted against x on log-log paper (Fig. 1b), Equation 4 produces a straight line with b as the slope. If $\log y$ is plotted against $\log x$, the x -intercept

(where $\log x = 0$) is $\log a$. Accordingly, if y is plotted against x on log-log paper, the line crosses $x = 1$ where $y = a$. Log-log plots with straight lines are common in geology textbooks. As a result, power functions arise often in geology.

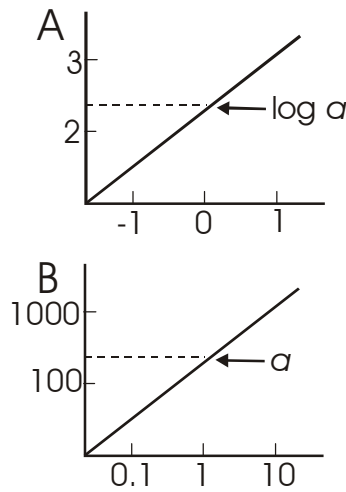


Figure 1. Power function plotted as $\log y$ vs. $\log x$ on arithmetic paper (A) and y vs. x on log-log paper (B).

The key points can be illustrated with the data in Table 1. The first column is a counter (the data are from a larger, published data set). The second and third columns, labeled "x" and "y" are the measured quantities, and the fourth and fifth column give the logarithms of the measured quantities.

	x	y	$\log x$	$\log y$	$y\text{-hat}$
692	0.16	0.61	-0.796	-0.215	0.527
691	0.54	0.98	-0.268	-0.009	1.012
689	2.3	1.7	0.362	0.230	2.203
693	7.8	3.85	0.892	0.585	4.243
694	9.6	5.1	0.982	0.708	4.743
690	18.2	6.6	1.260	0.820	6.687
695	19.4	8.3	1.288	0.919	6.920

Table 1. Data for Figures 2 and 3

The object now is to fit a straight line to the data in columns 4 and 5. Using the built-in linear-regression routine of an all-purpose spreadsheet program, I get output that includes the following. The "constant" is given as 0.148761; the "r-squared" value is 0.977223; and the "x-coefficient" is 0.53687. This means that the straight-line fit by this routine is:

$$\log y = 0.148761 + 0.53687 \log x . \tag{5a}$$

Taking the antilogarithms of both sides produces

$$y = 10^{0.148761} x^{0.53687}, \quad (5b)$$

or, after rounding,

$$y = 1.41x^{0.54}. \quad (5c)$$

The last column in Table 1 ("*y-hat*") gives the values "predicted" by applying Equation 5a to the data in column 4. Figure 2 shows the fitted line. The squares are the data of columns 2 and 3, and the line is *y-hat* vs. *x*.

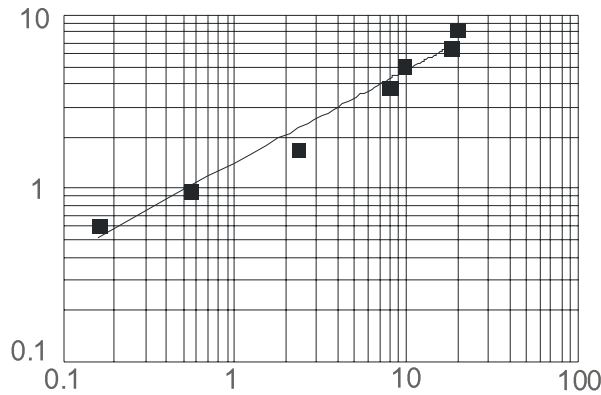


Figure 2. Graph of *y* vs. *x* from Table 1 on log-log paper. Note the axes.

(Without intending to go into least-squares regression, I should point out that the spreadsheet routine uses "ordinary least squares". The line is the one that minimizes the sum of the squares of the deviations of *y-hat* from *y*, and r^2 is the coefficient of determination, which would be 1.00 for a perfect straight line. For a discussion of this technique, see Taylor [1997]. For a discussion of pros and cons of the technique vs. alternatives for the kind of problem discussed in this chapter, see Williams and Troutman [1990], Smith [1994] and Nicklas, [1994]).

One gets so used to seeing power functions as log-log plots, it is worthwhile to consider how they look on normal, arithmetic graph paper. Figure 3 shows the information of Figure 2 on the normal graph paper. Notice that the curve, which represents Equation 5c, continues to increase, but it increases at an ever-decreasing slope. One can see this by taking the derivative of the function. From Equation 5c,

$$\frac{dy}{dx} = 0.76x^{-0.46} \quad (6)$$

Remembering that dy/dx is simply the *slope function* (see CG-5, "If Geology, Then Calculus," March 1999), Equation 6 demonstrates that the slope decreases continuously as *x* increases, because $x^{-0.46}$ is a *decreasing* function.

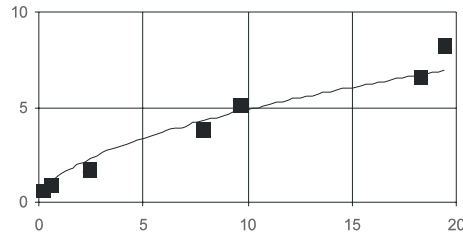


Figure 3. Graph of y vs. x from Table 1 using arithmetic coordinates. Compare the axes to those of Figure 2.

Beyond the particular case of Equation 5c, the shape of the power function in arithmetic coordinates can be inferred easily by differentiating the general form, Equation 3. Taking the derivative,

$$\frac{dy}{dx} = (ab)x^{b-1} . \quad (7)$$

If $b < 1$, then the exponent of Equation 7 is *negative*, which means that the slope, dy/dx , is a decreasing function as in Equation 6. On the other hand, if $b > 1$, then the exponent of Equation 7 is positive, and this means that dy/dx increases with increasing x . If $b = 1$, then the slope is constant ($dy/dx = a$). So, in summary, $y = ax^b$ plots as a convex-upward curve if $b < 1$, a straight line if $b = 1$, and a convex-downward curve if $b > 1$.

As pointed out in an earlier column (CG-3, "Progressing Geometrically," November 1998), a straight-line log-log plot of y vs. x is produced if one geometric progression (y -series) is paired against another geometric progression (x -series). Thus, one can "read" Equation 3 as follows: For every increase of x by some factor, y changes by some other factor. This other factor can be found easily from the slope, b . For example, in Equation 5c, if x increases by a factor of 2.0, then y increases by a factor of 1.45. To see this, write Equation 5c for x_1 and x_2 , let $x_2 = 2x_1$, and divide the two equations:

$$y_1 = 1.41x_1^{0.54} , \quad (8a)$$

$$y_2 = 1.41(2x_1)^{0.54} , \quad (8b)$$

$$y_2 / y_1 = 2^{0.54} = 1.45 . \quad (8c)$$

Similarly, a 3-fold increase in x causes y to increase by a factor of 1.8, and a 10× increase in x produces a 3.47× increase in y . Thus the ratio y/x changes as x changes if $b \neq 1$. That is why these functions plot as curves on arithmetic paper if $b \neq 1$.

The Function $y=b^x$ is Something Different

Many students confuse $y = x^b$ with $y = b^x$. The first is a power function, the subject of this chapter. The second is an exponential function, a general form of which is

$$y = ab^{kx} . \quad (9a)$$

The function is written with a general base, b , in Equation 9a. More commonly, it is written with the base e :

$$y = ae^{kx}. \quad (9b)$$

Equation 9b is the general form of "the" exponential function, the function that describes radioactive decay ($k < 0$) and unbridled geometric growth ($k > 0$).

One can see the connection between Equations 9a and 9b by recalling the definition of a logarithm (from CG-2 "Speaking Logarithmically," September 1998),

$$b = e^{\ln b}, \quad (9c)$$

and applying it to Equation 9a,

$$y = a(e^{\ln b})^{kx} = ae^{(k \ln b)x}. \quad (9d)$$

The result is

$$k' = k \ln b, \quad (9e)$$

which shows that Equations 9a and 9b are interchangeable. As illustration, $y = 10^x$, $y = e^{2.303x}$, and $y = 2^{3.322x}$ are all the same function. If you don't believe it, let $x = 2$ and calculate y each way.

Unlike a power function, which produces a straight-line graph of $\log y$ vs. $\log x$, an exponential function produces a straight-line graph of $\log y$ vs. x (i.e., a straight line on semi-log paper with y on the log scale). In other words, for an exponential function with $b > 0$, y increases by a factor every time x increases by an increment. For an exponential function with $b < 0$, successive incremental increases in x cause y to decrease successively by the same factor (see CG-3). That is what the concept of half-life is all about.

Hack's Law

As you have no doubt anticipated, the data of Table 1 are drawn from Hack (1957): x is drainage area (mi^2), and y is stream length (mi). Column 1 lists the site number. The stream length is the distance measured from the site along the bends and meanders to the drainage divide above the longest stream. The area is the drainage area above the site; it includes the drainage basin of the principal stream and all of the tributaries that enter above the site. Referring to the entire data set, Hack (1957, p. 47) said that stream lengths were measured on topographic maps and aerial photographs with a map measure and, in a few cases, in the field by tape traverse. Drainage area was measured by planimeter, generally on topographic maps, and in a few cases on aerial photographs.

The seven sites of Table 1 are all on Gillis Creek, a tributary of the Patapsco River, Maryland. Gillis Creek drains schists of the Appalachian Piedmont.

Hack (1957) and subsequent authors have commented on the fact that the exponent of the length-to-area relationship is larger than 0.5. The 0.5 value is a benchmark, because it is

consistent with geometric similarity. The relevance of geometric similarity can be seen as follows. Consider a rectangle with length, L , width, w , and area A . The area, of course, is,

$$A = wL . \quad (11a)$$

Now define a shape characteristic, s , by

$$s = w / L , \quad (11b)$$

the width-to-length ratio. Geometrically similar rectangles will all have the same width-length ratio; i.e., s will be constant for similar rectangles. So, combining Equations 11a and 11b, one can get L in terms of s and A :

$$w = sL ; \quad (11c)$$

$$A = sL^2 ; \quad (11d)$$

$$L = (1/s)A^{0.5} . \quad (11e)$$

Thus, geometrically similar rectangles produce a length-to-area power function with exponent of 0.5. Other shapes would do the same, provided they are geometrically similar as they change in size. Geometrically similar means that all length dimensions increase by the same factor as they change size.

That the exponent of Equation 5c is not 0.5 may be accounted for if the drainage basin of Gillis Creek does not maintain geometric similarity as one moves from site to site. Hack (1957) showed this (for Equation 1) with an argument that goes as follows. First define the average width, w_a of the drainage basin:

$$w_a = A / L . \quad (12a)$$

Then combine Equations 5c and 12a:

$$L = 1.41(w_a L)^{0.54} . \quad (12b)$$

Raise each side to the 1/0.54 power:

$$L^{1.85} = 1.89w_a L . \quad (12c)$$

Divide each side by L^2 :

$$L^{1.85} / L^2 = 1.89w_a / L . \quad (12d)$$

Rearrange:

$$w_a / L = 0.53L^{-0.15} . \quad (12e)$$

This result shows that the width-to-length ratio of the drainage basin – the shape factor – decreases as one goes downstream in Gillis Creek. The basin does not maintain geometric similarity. For the overall data set (Equation 1), Hack obtained

$$w_a / L = 0.57L^{-0.33} . \quad (13)$$

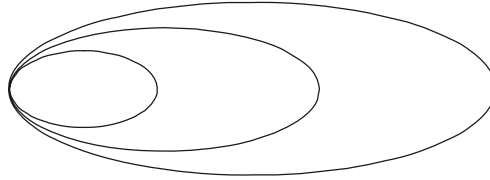


Figure 4. Nested dissimilar ellipses. Long axis increases geometrically by a factor of 2, while short axis increases geometrically by a factor of 1.25.

The downstream decrease in the width-to-length ratio (Equation 12e) means that the basin becomes more elongate downstream (e.g., see Fig. 4). The elongation can be appreciated more directly by multiplying Equation 12e by L to obtain

$$w_a = 0.53L^{0.85} . \quad (14)$$

By the reasoning that we have already discussed in some detail, Equation 14 indicates that w_a vs. L on arithmetic graph paper would be a convex-upward plot. The geometric increase in w_a does not keep up with the geometric increase of L . For example, if the basin doubles in length, then its average width increases by a factor of only 1.8; similarly, a 10-fold increase in L produces only a 7.1-fold increase in w_a . (From equation 13, w_a increases 1.6× and 4.7× for an increase in L of 2× and 10×, respectively, in Hack's full data set.)

Over the years, there has been quite a bit of discussion of the exponent in Equation 1 and the notion that it might imply that larger basins are more elongate. Hack's data set, which was near Washington D.C., consisted of about 90 sites, and A ranged through four orders of magnitude, from 0.03 to 300 mi². He said the same line (Equation 1) also passes through the log-log plot of L vs. A from about 400 U.S.G.S. stations in the northeastern U.S. Gray (1961) used topographic data from 47 small watersheds in the Midwest and North Carolina supplemented by a couple dozen sites from the literature for North and Middle Atlantic States and came up with

$$L = 1.40A^{0.568} , \quad (15a)$$

for A ranging from 0.3 to 2000 mi². Mueller (1973) compiled "thousands" of length-area pairs from around the world; randomly selected 250 of them ranging from 0.01 to 3,000,000 mi² (8 orders of magnitude); and found the whole lot to be described by

$$L = 1.6386A^{0.5536} . \quad (15b)$$

On the other hand, Mueller also noted that, over the huge range of his data, there appeared to be three segments. From 0.01 to 8,000 mi², Hack's original equation worked "extremely well". But from 8,000 to 100,000 mi², an exponent of 0.5 was the best fit, and, for $A > 100,000$ mi², he found an exponent of 0.466. More recently, Montgomery and Dietrich (1992) combined their own data with many published sets (including Hack's, Gray's, and Mueller's) to get a data set spanning (p. 827) "more than 11 orders of magnitude in basin area, from unchanneled hillside depressions to the world's largest rivers". They found the variation to be reasonably fit by

$$L = (3A)^{0.5}. \quad (16)$$

Theoretically, it is possible for the exponent of Hack's law (b) to be larger than 0.5 and not imply basin elongation with increasing A . This is so because Hack, and some of the others, measured L along the sinuous course of the river. Early on, Smart and Surkan (1967) showed that variation in sinuosity downstream (increasing A) could account for the departure of b from 0.5 in geometrically similar basins. More recently, Mandelbrot (1983) and Peckham (1995) developed schemes for $b > 0.5$ with geometrically similar basins. Peckham (1995), in particular, noted that the L of Equation 16 is a straight-line basin length and so took Montgomery and Dietrich's finding to be part of the "mounting evidence" that basin elongation is not the cause of Hack's law.

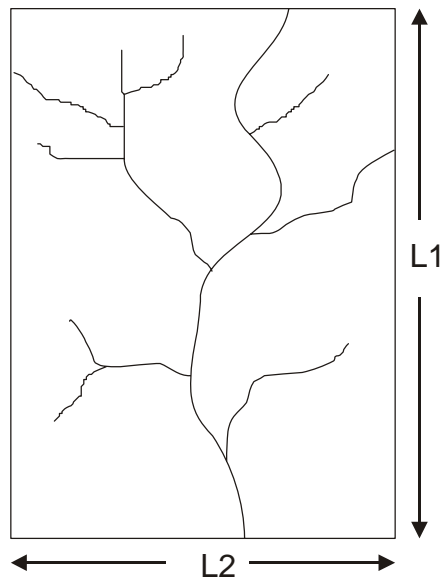


Figure 5. Stream network enclosed by rectangle.

Rigon et al. (1996) pulled the threads together with a study distinguishing between stream lengths (L) and basin lengths. They defined the long dimension (L_1) and short dimension (L_2) of basins by enclosing them in rectangles (Fig. 5). Using U.S.G.S. digital elevation models, they gathered data for 21 basins, mainly in West Virginia and Kentucky, over a range of areas from 20 to 800 mi². They found the following relations:

$$L \propto A^{0.6}; \quad (17a)$$

$$L_1 \propto A^{0.52}; \quad (17b)$$

$$L \propto L_1^{1.15}; \quad (17c)$$

$$\text{and } L_1 / L_2 \propto L_1^{-0.08}. \quad (17d)$$

Thus they found a Hack-type exponent in basins that become slightly more elongate with increasing size, but the variation in sinuosity (L/L_1) has a larger effect on b .

In the introduction of their paper, Rigon et al. (1993, p. 3367) wrote (p. 3367),

... Although the exponent in the power law may slightly vary from region to region, it is generally accepted to be slightly below 0.6. Equation 1, rewritten as $L \propto A^h$ with $h > 0.5$ is usually termed "Hack's law."

Hack's equation has become a law.

Isometry, Allometry and Scaling

In the morphometrics of biology, the question of shape changing with size is the distinction between isometry and allometry, and Equation 3 is known as the allometric equation. The language originated with Sir Julian Huxley (1887-1975), renowned evolutionist and zoology professor at King's College, University of London. The analog of Hack's Appalachian drainage basins is the fiddler crab (*Uca pugnax*).

The male fiddler crab has one claw very much larger than the other. The female fiddler crab has two small, equal-size claws, which are the same size as the small claw of the male. As the male fiddler grows, its larger claw grows more than its body. The large claw becomes increasingly more out of proportion with its body. With increasing size, therefore, the animal changes shape.

On a visit to the marine laboratory at Woods Hole, Huxley investigated the changing relative size of the fiddler's large claw (Huxley, 1924). He weighed 400 crabs, and he then weighed their large claw. He let y be the weight of the claw and x be the weight of the rest of the body. From a plot of $\log y$ vs. $\log x$, he found a "remarkably straight line" for crabs with total weight between 60 mg and >3.5 g. Actually, he found two successive straight lines: a slope of 1.61 to 1.64 for crabs whose large claw weighed less than about 30% of the total weight, and a slope of 1.32 to 1.35 for larger crabs. He attributed the change in slope to the crabs' reaching sexual maturity. For females, he reported that the weight of a claw is a constant 2% of the weight of the animal, and that the same is true for the small claw of the male.

After a variety of terms had been used by various workers on disproportionate growth, Huxley and Tessier (1936) introduced the modern language. They proposed *allometry* for "growth of a part at a different rate from that of the body as a whole or of a standard" and *isometry* for "the special case where the growth rate of the part is identical with that of the standard or whole" (p. 780). *Positive* allometry occurs when the part grows faster than the whole; *negative* allometry occurs when the part lags behind. The large claw of the male fiddler crab illustrates positive allometry. The post-birth human head is a familiar example of negative allometry. The small claw of the male fiddler crab is an example of isometry.

For the allometric equation, Huxley (1924) started with $y = bx^k$. Other workers took issue with " k " being used for the exponent, because it is not necessarily constant (as shown by

the two values of Huxley's crab claws). Huxley and Tessier (1936) changed the k to α , so that a Greek letter would stand in contrast to the other, Latin letters. The equation, $y = bx^\alpha$ is widespread in the biological literature of allometry, as is discussion of the meaning of b and α (e.g., Gould, 1966). I am sticking with the more general $y = ax^b$, which seems preferable for cross-disciplinary application (e.g., Berry et al., 1989, p. 121; Hastings and Sugihara, 1993, p. 7). The important book on allometry by Schmidt-Nielson (1984), it should be noted, uses $y = ax^b$.

The "standard" referred to in Huxley's definitions is isometry, or geometric similarity. If x and y are dimensionally the same, geometric similarity dictates that the exponent, b , of Equation 3 must be 1. This is the case for Huxley's study, because both x and y were weights. Thus, for the large claw of Huxley's juvenile male crabs,

$$y \propto x^{1.62} \tag{18}$$

signifies allometric growth, whereas the relation for the claws of the female crabs,

$$y \propto x^1, \tag{19}$$

is isometry.

But, if x and y are not dimensionally the same, geometric similarity dictates another value for b . For example, if Huxley had chosen to use y for the *weight* of the claw and x for the *length* of some other body part, then the isometric relation for the female claw would be

$$y \propto x^3, \tag{20}$$

because weight is proportional to volume (assuming uniform density) and volume dimensionally is L^3 . Thus, for this different choice of x and y , Huxley would have found

$$y \propto x^{4.86} \tag{21}$$

for the disproportionate claw of the male crab (assuming all densities are the same). Similarly, if Huxley had used length for y and weight for x , the isometric standard would be

$$y \propto x^{0.33}, \tag{22}$$

and the male claw would be

$$y \propto x^{0.54} . \tag{23}$$

This is basically the same argument used in Equations 11 to show that geometrically similar (i.e., isometric) drainage basins should produce a Hack-type exponent of 0.5.

The argument we have just worked through for an isometric fiddler crab is an example of the deductive approach to scaling. Scaling, in general, relates to changes in size. In biology, scaling "deals with the structural and functional consequences of changes in size or

scale among otherwise similar organisms" (Schmidt-Nielson, 1984, p. 7). The deductive, or engineering, approach deals explicitly with dimensions. Such analysis is one of the many brilliant insights of Galileo (1564-1642). He wrote it out in the form of a three-way conversation (*Dialogues Concerning Two New Sciences*, 1638) while under house arrest at the end of his life for arguing convincingly, in another conversation between the same players (*Dialogues Concerning the Two Chief World Systems*, 1632), that the Earth revolves around the Sun.

The argument that Salviato presents to Sagredo and Simplicio (in Day Two of *Two New Sciences*) is that weight scales as L^3 and resistance to crumbling under the load scales as L^2 . Therefore, as the size increases, the ratio of weight to resistance increases. Then, Salviato goes on to conclude:

From what has already been demonstrated, you can plainly see the impossibility of increasing the size of structures to vast dimensions either in art or in nature....

.... if one wishes to maintain in a great giant the same proportion of limb as that found in an ordinary man he must either find a harder and stronger material for making the bones, or he must admit a diminution of strength in comparison with men of medium stature; for if his height be increased inordinately he will fall and be crushed under his own weight.... Thus a small dog could probably carry on his back two or three dogs of his own size but I believe that a horse could not carry even one his own size."

Had Galileo known the word, Salviato would have said that the dog-to-horse scaling needs to be allometric, if the horse is to carry another horse.

Can we use Galileo's line of thinking to predict the allometry of animals whose skeletons are sturdy enough to hold up their weight? The book by Schmidt-Nielson (1984, p. 44) lays out the argument basically as follows. In order to maintain a constant ratio of body mass (M_b) to cross-sectional area of skeletal bones (A_s), skeletal areas must scale to body mass as:

$$A_s \propto M_b \quad . \quad (24a)$$

But body mass is proportional to volume, and volume scales as L^3 . Therefore, the length of skeletal bones would scale with body mass as:

$$L_s \propto M_b^{0.333} \quad . \quad (24b)$$

The skeletal mass (M_s) is proportional to skeletal volume, and the skeletal volume goes as $A_s L_s$. Therefore,

$$M_s \propto M_b^{1.333} \quad , \quad (24c)$$

by multiplying Equations 24a and 24b together. According to Equation 24c, an animal that weighs twice as much as another animal should have a skeletal mass 2.5× as much; another animal that weighs 10× as much, should have a 21× more massive skeleton.

At first glance, these numbers may seem reasonable, but one can easily show that they are not. A marmoset, at the small end of the size range of primates, has an M_b of about 100 g (Fleagle, 1985, Fig. 5). Its M_s , let's say, is 5 g or 5% of its body weight (I'm guessing; where I get that number will be clear shortly). The gorilla is at the other end of primate sizes. A 200-kg gorilla would weigh 2000× as much as the marmoset. According to Equation 24c, the gorilla's M_s would be 25,000 (i.e., $2000^{1.333}$) times that of the marmoset. Thus the gorilla's skeleton would be about 120 kg, or some 60% of its body weight. Now, I don't know anything about gorillas, but that seems pretty high.

To see that it really is too high, consider a 2500-kg elephant, which is 25,000× heavier than the marmoset. Its M_s would be $25,000^{1.333}$ or about 730,000× times heavier than the marmoset's. This works out to be 3600 kg, more than the whole elephant. Clearly, something is wrong.

Papers by Medawar (1945) and Gould (1966) draw a distinction between the deductive and empirical approach to such scaling problems. The empirical approach uses actual data. What do the data say?

The book by Schmidt-Nielson (1984) shows a log-log plot of M_s vs. M_b for mammals and gives the empirical relation:

$$M_s = 0.061M_b^{1.09}, \quad (25)$$

(which is the source of the marmoset's M_s that I have been using). Using this relationship, our gorilla's M_s would be about 4000× that of the marmoset and be only about 10% of its own M_b ; our elephant's M_s would be about 60,000× more than the marmoset's and be about 12% of its M_b . Clearly M_s is allometric with respect to M_b , but not as much as considerations of static loading would make one think.

The rest of the story, according to Schmidt-Nielson (1984, p. 46), includes stresses on the bones of "hopping kangaroos, jumping dogs, galloping antelopes and buffalo, and running elephants" (not to mention scurrying marmosets). With respect to standing loads, the smaller animals are more "overdesigned" (have a larger "safety factor") than the larger ones, and so the actual slope on the log-log plot is less than that of engineering for static loads.

You have probably noticed that this last example, the M_s vs. M_b of mammals, is a very different flavor of allometry than the example of Huxley's fiddler crabs. The fiddler crabs illustrate what Gould (1966) labeled *ontogenetic allometry*: a variation that reflects growth over an organism's life history (ontogeny). As Gould noted, there are two ways of approaching it: directly with longitudinal studies (where identified organisms are traced through their growth stages); and indirectly, where many individuals at different growth stages are measured at the same time (e.g., Huxley's crabs). The other example illustrates *interspecific allometry*: a variation between species within a larger taxonomic group and all at the same growth stage (generally adults). Graphs of interspecific allometry are referred to as "mouse-to-elephant plots" by Gould and other allometrists. They are like the rills-to-Amazon log-log plot discussed under Hack's law. Another type of allometry -- not discussed here, but obviously directly paleontological -- is evolutionary allometry: a variation between species within a phylogenetic lineage (Gould, 1966).

Huxley's crabs and our marmoset-to-gorilla argument illustrate mass-to-mass relationships, but there are many other combinations of variables. Among the many

interspecific relations in Schmidt-Nielson (1984) are such things as the volume of lungs (V_L , in liters) vs. body mass (M_b , kg) for 21 mammals from bats to whales,

$$V_L = 0.05M_b^{1.02}, \quad (26a)$$

and the surface area of gills (A_g , m²) vs. M_b (kg) for 19 species of fish,

$$A_g = 0.40M_b^{0.82}. \quad (26b)$$

The same analysis can be done for plants, of course. Many such relations are discussed by Niklas (1994), including, for example, the leaf area (A_L , cm²) vs. stem diameter (D , cm) for 46 North American deciduous trees,

$$A_L = 917D^{1.84}, \quad (27a)$$

canopy spread (L_s , m) vs. height (H , m) for record specimens of angiosperm trees,

$$L_s = 1.19H^{0.877}, \quad (27b)$$

and height (H , m) vs. trunk diameter (D , m) for 480 woody (non-palm) trees,

$$H = 20.6D^{0.535}. \quad (27c)$$

Power-function scaling has spread well beyond the geometric variables. By way of illustration, and sticking with mammals, here are a few more examples of empirical, mouse-to-elephant functions of M_b (kg) from Schmidt-Nielson (1984): metabolic rate (kcal/day)

$$P_{met} = 73.3M_b^{0.74}, \quad (28a)$$

rate of oxygen consumption while running (mL/sec)

$$P_{O_2} = 1.92M_b^{0.809}, \quad (28b)$$

heartbeat frequency (min⁻¹)

$$f_{heart} = 241M_b^{-0.25}, \quad (28c)$$

and longevity (yr)

$$t_{life} = 11.8M_b^{0.20}. \quad (28d)$$

(From Schmidt Nielson [p. 146]: "An alert reader may already have calculated that he may be dead.... Luckily we live several times as long as our body size suggests we should.")

Combining empirically determined relationships such as Equations 26-28 with the engineering approach to scaling gives enormous insight to biologic form and process, and this in turn informs paleobiology. I am a nonbiological geologist, and I found the books by Schmidt-Nielsen (1984) and Niklas (1994) to be absolutely fascinating. I had often wondered how paleobiologists could know so much about extinct animals' soft parts and physiology, for example, by working with only some preserved hard parts.

Stokes' Law

Stokes Law (Equation 2) states how the settling velocity of small spheres scales with size. Expressed as a power function, Stokes' Law is

$$v = A_s d^2, \quad (29)$$

where A_s is a parameter combining relevant properties of the particle and fluid with a shape-specific coefficient (1/18). Given that Stokes derived Equation 2 from first principles, we can say that the power function (Equation 29) was obtained via the engineering approach to scaling. The equation predicts that, so long as sedimentary particles are small spheres, a log-log plot of settling velocity vs. one-dimensional size (diameter, radius, or circumference) will produce a straight line with a slope of 2.

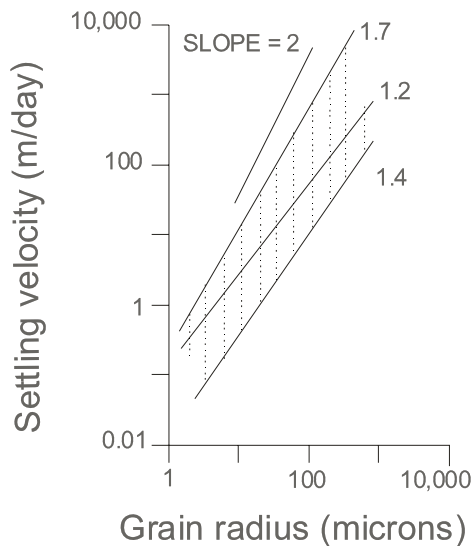


Figure 6. Graph of settling vs. grain radius for small biogenic particles. Foraminifera and radiolaria plot along the upper line ($b=2$). Diatoms and flagellates plot along the dashed line ($b=1.2$). Adapted from Lerman (1979, Fig. 6.3).

Figure 6 summarizes a "mouse-to-elephant plot" Lerman (1979) for the settling velocity of small biogenic particles. Foraminifera and radiolaria comply with the predicted slope. Living and dead cells of flagellates and diatoms fall along a line with a slope of 1.2 within a shaded band bounded by lines with slopes of 1.4 and 1.7. Thus for these particles, Stokes' Law for spheres does not hold.

As Lerman (1979) noted, the departure of the actual settling velocities from the predicted ones can be explained if the bulk density of the particle changes with size. In particular, if the bulk density decreases as particle size increases, then the departure would increase with increasing size as shown in Figure 6.

Bulk density would decrease with increasing size if the particles were hollow spheres, rather than solid ones. One can easily see this by dusting off a little bit of geometry, as follows. The surface area (S_s) and volume (V_s) of a sphere are:

$$S_s = 4\pi r^2, \quad (30a)$$

$$\text{and } V_s = (4/3)\pi r^3, \quad (30b)$$

respectively. Replacing the surface with a thin shell of thickness Δr , the mass of the shell (M_{shell}) is:

$$M_{shell} = 4\pi\rho_s r^2 \Delta r, \quad (30c)$$

where ρ_s is the density of the solid material composing the shell. On the other hand, the bulk density, which counts the volume occupied by the whole spherical shell, is the ratio of M_{shell} to V_s :

$$\rho_b = 3\rho_s \Delta r / r. \quad (30d)$$

from Equations 30b and 30c. Therefore, the density ratio for hollow vs. solid spheres is

$$\rho_b / \rho_s = 3\Delta r / r, \quad (30e)$$

from Equation 30d. This result shows that, if the shell thickness stays the *same percentage* of the sphere diameter as the shell increases in size, the bulk density of the shell is independent of size. But, if the shell thickness stays *the same number of microns* as the shell increases in diameter, the bulk density decreases as r^{-1} .

In the derivation of Equations 30, we did not consider the fluid inside the sphere. Effectively, we gave it a zero density. Lerman (1979) did the more complicated derivation that involves the internal fluid. Giving it a density of ρ_f , the same as the fluid that surrounds the shell, he obtained

$$(\rho_b - \rho_f) = (\rho_s - \rho_f)(3\Delta r / r). \quad (31a)$$

This result provides a direct bridge to Stokes' Law. The A_s of Equation 29 is:

$$A_s = \frac{(\rho_s - \rho_f)g}{18\mu}, \quad (31b)$$

where ρ_s the density of the solid sphere. Another way of looking at ρ_s of Equation 31b is that it is the density of a sphere consisting of material of density ρ_s . But this is the same meaning

as ρ_b of Equation 31a. So the factor $(\rho_s - \rho_f)$ of Equation 31b can be replaced by the entire right side of Equation 31a to produce the A -parameter for the spherical shell:

$$A_{shell} = \frac{(\rho_s - \rho_f)g}{18\mu}(3\Delta r / r), \quad (31c)$$

where ρ_s is the density of the material composing the shell. The numerical value of all these ρ_s 's is the same, so Equations 31b and 31c can be combined to produce

$$A_{shell} = (3\Delta r / r)A_s. \quad (31d)$$

Thus we have

$$A_{shell} \propto r^{-1}. \quad (31e)$$

Combining this result with

$$v_{shell} = A_{shell}d^2 \quad (31f)$$

and

$$d^2 \propto r^2 \quad (31g)$$

produces

$$v_{shell} \propto r^1. \quad (31h)$$

So, the settling velocity of hollow shells with a fixed shell thickness would plot with a slope of 1 on a log-log plot of settling velocity vs. diameter.

As succinctly stated by Lerman (1979, p. 424), "The settling velocities of certain planktonic organisms in seawater ... are functions of a power of r that lies between 1 and 2. Such settling velocities fall in the range between the hollow sphere and solid sphere Stokes' equations." The argument behind this statement is another example of combining the engineering and empirical approaches to a scaling problem.

Final Remarks

The power function is a springboard to understanding many geological phenomena. Power laws, self-similarity and nonintegral dimensions are all key concepts in fractal geometry, probably the hottest topic of the 1990's in geomathematics. The papers by Peckham (1995) and Rigon et al. (1996) take a fractal approach to Hack's law, and drainage nets are examined in many books on fractals (e.g., Mandelbrot, 1983; Korvan, 1992; Turcotte, 1997; Rodriguez-Iturbe and Rinaldo, 1997). Measuring the length of a sinuous river on a map with a map measure is reminiscent of Mandelbrot's famous question, "How long is the coast of Britain?"

A hot topic in evolution now is heterochrony, the process that connects genetics to natural selection (McKinney and McNamara, 1991; McNamara, 1997). Underlying the idea is the differential form of Equation 3 for ontogenetic allometry,

$$\frac{dy/dt}{y} = b \frac{dx/dt}{x}, \quad (32)$$

(divide Equation 7 by Equation 3) and the question of what happens if a heritable mutation causes a change in the ratio of $(dy/dt)/y$ to $(dx/dt)/x$ that occurs during ontogeny. A key paper in the development of the idea was entitled "Allometry in ontogeny and phylogeny" (Gould, 1966).

Rather than Stokes' Law, which is limited to a small region of small diameters or low fluid viscosities, the equation of choice now for studies of settling velocity as a function of grain diameter is the Gibbs equation for spheres with the density of quartz (Gibbs et al., 1971). This is not an *a priori* power function, but rather a polynomial found by fitting mountains of settling-tube data. But, interestingly, empirical power functions are used to correct for deviations from the Gibbs equation resulting from differences in grain shape (Komar et al., 1984) and to make the connection from sieve data to settling velocities (Baba and Komar, 1981).

There is no escaping the power function.

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