Journal of Geoscience Education, v. 47, n. 4, p. 376-383, September 1999 (edits, June 2005)

Computational Geology 7

The Algebra of Unit Conversions

H.L. Vacher, Department of Geology, University of South Florida, 4202 E. Fowler Ave., Tampa FL 33620

Introduction

The Computational Geology column of the last issue concerned problem solving. We worked through three elementary geological-mathematical word problems in some detail. Although these problems were not designed to bring up the subject of unit conversions, it arose anyway. The fact is, it is difficult to consider any interesting problem without having to convert units.

In my experience, nothing gives students more trouble than unit conversions. More errors seem to be made in unit conversions than in any other part of geological-mathematical problem solving. These errors can produce outlandish results. For example, some students calculate that groundwater sometimes travels at the speed of sound, or that the water table in Florida can be found at an elevation exceeding that of Mt. Everest. And then they get annoyed with me for not giving them partial credit for such answers! My position on the matter is that there is no point in knowing the right equation to use if you cannot deal properly with the numbers that go into it or come out of it. In other words, you will not get off square one in geological-mathematical problem solving if you cannot convert units, reliably without error.

My experience is by no means unique. One can get a sense of the ubiquity of unit-conversion difficulties in college classrooms from the number of course-specific Web sites that address the subject. They are particularly prevalent at sites associated with physics, chemistry, astronomy and engineering courses, but they are also beginning to appear in Web materials of introductory geoscience courses as these courses become more quantitative.

No one should be surprised, however, to hear that science professors throughout the country find that students have difficulty with unit conversions. Unit conversions are simply an exercise in applied algebra. People coming out of high school today are *notoriously poor* at algebra. It is *well known* now that, statistically, they are among the world's worst at algebra. More about that later.

About Algebra

In 1993, there was a conference in Washington D.C. on reforming the learning and teaching of algebra from kindergarten through graduate school. Called "The Algebra Initiative Colloquium", this conference resulted in a two-volume set of papers (Lacampagne et. al, 1993). The lead paper, the text of the keynote talk, was by Victor J. Katz, author of a popular textbook on the history of mathematics. Katz started his talk with the question, What is algebra? He noted that there is usually no definition in twentieth century texts. It is apparently assumed that everyone knows what algebra is.

Intrigued by the question, I consulted a mathematical dictionary. I was told that algebra is the branch of mathematics that deals with the general properties of numbers and the generalizations arising therefrom. That definition didn't do much for me.

I asked my Computational Geology class the same question. Without hesitation and with universal agreement, the class responded with, "It's finding unknowns. It's solving equations".

According to definitions given in Katz's paper, my class is right in line with eighteenth century textbooks (which is fine with me). According to Maclaurin's *A Treatise of Algebra in Three Parts* (1748, p. 1), algebra is "a general method of computation by certain signs and symbols which have been contrived for this purpose and found convenient." According to Euler's *Complete Introduction to Algebra* (1767, p. 186), "Algebra has been defined as the science which teaches how to determine unknown quantities by means of those that are known." (Quotations are from Katz's article, "The development of algebra and algebra education", in Lacampagne et. al, 1993).

There was a long history of algebra before Colin Maclaurin (1698-1746) and Leonhard Euler (1707-1783). The history goes back to before there were symbols to manipulate. The earliest text, according to Katz, was by Muhammad ibn Musa al-Khwarizmi (*c*. 780-850) of ninth century Baghdad: *The Condensed Book on the Calculation of al-Jabr and al-Muqabala*. For a definition of Islamic al-jabr and al-muqabala, Katz used the writings of al-Khayyami (1048-1131; better known in the West as Omar Khayyam, author of the *Rubaiyat*): "One of the branches of knowledge needed in that division of philosophy known as mathematics is the science of al-jabr and al-muqabala which aims at the determination of numerical and geometric unknowns."

"Algebra", of course, is derived from "*al-jabr*". As noted by Katz, algebra has been about solving equations right from the first use of the word. The task itself is much older. Katz spoke of the algebra in the Rhind papyrus of Egypt, about 1650 B.C.

Today, our algebra of finding unknowns by manipulating equations is commonly called *school algebra* by mathematicians. This differentiates it from modern or abstract algebra.

Using *Al-Jabr*

Al-jabr and *al-muqabala* of the Islamic writers referred to specific operations. *Al-jabr* was the task performed to go from the first to the second of these equations (these are Katz's examples):

$$3x + 2 = 4 - 2x, \qquad (1)$$
$$5x + 2 = 4 . \qquad (2)$$

Al-jabr, therefore, was the transposition of a subtracted quantity from one side to the other side of an equation by adding it to both sides. *Al-muqabala* was the task that changes Equation 2 to:

$$5x = 2 . \qquad (3)$$

Thus *al-muqabala* was performed by subtraction of equal amounts from both sides to reduce a positive term. Note that *al-jabr* and *al-muqabala* are examples of the Fundamental Commandment of School Algebra (FCOSA): Do to both sides equally.

Of particular interest to me a thousand years later is that these algebraic operations were not performed with equations and symbols. The textbooks of al-Khwarizmi and Omar Khayyam consisted of step-by-step instructions, in words, of how to solve particular problems. For example, in the spirit of these Islamic mathematicians, here is how a classroom problem of today would be treated in the so-called *rhetorical algebra* that they used:

> Suppose the temperature is 20°C. What is the temperature in degrees Fahrenheit? The solution is this: You take the 20. This you multiply by nine; the product is 180. Divide this by five; the result is 36. To this you add 32; the sum is 68. This is the temperature in degrees Fahrenheit that you sought.

Here is an example from a geology classroom:

> Suppose the tectonic plate is moving at 5 cm/yr. What is this speed in miles per millions of years (mi/m.y.)? The solution is this: You take the 5 cm/yr. This you divide by 100,000 cm/km; the result is 0.00005 km/yr. Multiply this by 1,000,000 years per m.y.; the product is 50 km/m.y. Divide this by 1.6 km/mi; the product is 30 mi/m.y. This is the speed of the tectonic plate in the units that you sought.

These sets of instructions are *algorithms*. An algorithm is a step-by-step procedure for solving a problem. Appropriately, the word has a connection to al-Khwarizmi. It comes from *Algoritmi dixit*, the first words of *Liber algoritmi de numero Indorum* (generally referred to as *Liber algorismi*), the Latin translation of al-Khwarizmi's text on arithmetic. The *Liber algorismi* is the work that brought our Hindu-Arabic number system to the West. *Algortmi dixit* means "al-Khwarizmi says".

Al-Jabr **and Unit Conversions**

These two examples of rhetorical algebra involve unit conversions, the subject of this column. No one would expect to use a manual written in that style anymore. Nevertheless, it is very similar to an approach that is commonly used for unit conversions. Many conversion tables, especially in engineering references, are written in the style of Table 1. Such tables have columns indicating (1) the target units, (2) the given units, and (3) the conversion factor.

Conversion Number	To obtain the number of:	Multiply the number of:	By:
1	centimeters	meters	100
2.	millimeters	meters	1000
3	microns	millimeters	1000
4	meters	kilometers	1000
5	feet	meters	3.281
6	meters	feet	0.3048
7	centimeters	feet	30.48
8	kilometers	miles	1.609
9	miles	kilometers	0.6214
10	feet	miles	5280
11	centimeters	inches	2.54
12	inches	centimeters	0.3937

Table 1. Conversion table for some lengths

To see the similarity to *Algortmi dixit*, consider conversion number 1. The line in the table says, "To obtain the number of centimeters, take the number of meters and multiply by 100."

Such tables are straightforward and easy to use. Their main drawback is finding them when you need them. Fortunately, you don't have to.

Symbolic Algebra

Table 1 differs from the two examples of rhetorical algebra in a fundamental way. It speaks of meters and centimeters, in general, not of particular numbers of them. The early style – not only of the Islamic scholars but also the author of the Rhind papyrus – was to give examples. It was up to readers to generalize. Given a new problem, the reader was supposed to apply the same algorithm, starting with a new number and tracing out a new succession of numbers through the cookbook procedure.

A huge step was made when the numbers of particular examples were replaced by symbols. Rhetorical algebra was replaced by *symbolic algebra*, in which symbols are used in place of numbers. The initiation of symbolic algebra was one of the mathematical achievements of Renaissance Europe. One of the important innovators was the Frenchman, Fraçois Viète (1540-1603), a contemporary of Galileo (1564-1642). Viète, whose main job was a lawyer and legal counselor for French kings, is considered by mathematics historians to be the central figure in the transition of algebra from an assortment of algorithms to an "analytical art".

To illustrate early symbolic algebra, here is the way that Viète would write an equation in his treatises, which are collectively known as *The Analytical Art*, at the turn of the seventeenth century:

$$1QC - 15QQ + 85C - 225Q + 274N \text{ aequatur } 120 . \qquad (4a)$$

In this equation "aequatur" means "equals"; N means "numero" (number; unknown); Q is N multiplied by itself (N-squared); C is N-cubed. In modern symbols, this equation would be written

$$x^5 - 15x^4 + 85x^3 - 225x^2 + 274x = 120. \qquad (4b)$$

It was the great French mathematician René Descartes (1596-1650) who introduced the convention of using letters near the end of the alphabet for unknowns and attaching to them superscripts of Hindu-Arabic integers to indicate the number of times they are multiplied together.

Viète also was the first to use letters to represent general coefficients. Descartes adopted the plan of using letters at the beginning of the alphabet for these general coefficients. John Wallis (1616-1703), an influential professor at Oxford, came up with fractional exponents. By the time of his writings at the end of the seventeenth century, the general form of equation 4 would be written

$$x^5 - ax^4 + bx^3 + cxx + dx = 0 . \qquad (4c)$$

Stating an equation in a form such as 4c allows it to be studied in generality. Symbols give immense power to algebra.

Equations of Unit Conversions

The equations that are used in converting units are not as complicated as Equation 4. For example, the conversion from Centigrade to Fahrenheit illustrated in the first example of rhetorical algebra is done by

$$F = 9C/5 + 32 \ , \tag{5}$$

where F indicates the number of degrees Fahrenheit and C represents the number of degrees Centigrade. To use this equation, you take the number of degrees Centigrade, substitute that number for C, and perform the operations indicated on the right side of the equation; the result is the number of degrees Fahrenheit. This step-by-step procedure is exactly the same as that spelled out in the rhetorical example. Letting $C = 20°$, one calculates that $F = 68°$.

An obvious benefit to the symbolic style is that it is evident how to go the other way, for example, from degrees Fahrenheit to degrees Centigrade. One does this by applying the FCOSA. First, subtract 32 from each side:

$$9C/5 = F - 32 \qquad .$$

Then multiply each side by 5/9:

$$C = 5F/9 - 17.78 \ . \tag{6}$$

For example, given that $F = 68^o$, C comes out to be 20^o.

The equations converting between Centigrade to Fahrenheit have the form:

$$y = ax + b, \tag{7}$$

which is the familiar linear function. The equation that applies to most unit conversions is even simpler:

$$y = ax, \tag{8}$$

which says that one unit is simply a multiple of the other.

For example, consider the first line of Table 1: "To obtain the length in centimeters, multiply the length in meters by 100". Clearly, this means that a meter is 100 times longer than a centimeter. The appropriate equation relating them, therefore, is:

$$1 \text{ m} = 100 \text{ cm}. \tag{9}$$

It is crucial that you notice that there is a big difference between Equations 5 and 9, not including their difference in form. The difference is in what has been substituted for the x's and y's. In the case of Equation 5, the x and y of the general form (Equation 7) are replaced by the *number* of degrees. In the case of Equation 9, the *x* and *y* of the general form (Equation 8) are replaced by *actual lengths* (x = 1 cm, and y = 1 m). The *F* and *C* of Equation 5 are *variables*, meaning they can be replaced by any numbers. The lengths indicated by letters in Equation 9 are not variables; they are fixed, one as a standard length, and the other as a defined fraction of that standard length. Equation 9 says that the *size* of the meter is 100 times the *size* of the centimeter. Equation 5 does *not* say that the size of a degree F is 32° larger than nine-fifths the size of a degree C.

The difference between Equations 5 and 9 is sufficiently important that it is worth belaboring a little. Look at what happens if you think of the Equation 9 as $m = 100c$, where *m* is the number of meters, and *c* is the number of centimeters. Suppose you measure a length to be 20 cm, and you want to convert that measurement to meters. Plugging the 20 into $100c$ produces $m = 2\,000$. So, you would be converting a length of 20 cm (less than a foot) to 2 km (more than a mile). That would certainly make a big difference in the answer to whatever question necessitated the conversion of centimeters to meters.

You can avoid this kind of blunder by never omitting the "1" in a unit-conversion equation when the equation refers to *sizes* of the units (which is nearly all the cases). Table 2 lists some common length conversions in equation form. Some are translations from specific lines of Table 1.

Conversion Number				
1	1 m	=	100 cm	
3	1 mm	=	1000 µm	
5	1 m	=	3.281 ft	
6	1 ft	=	0.3048 m	
7	1 ft	=	30.48 cm	
8	1 mi	=	1.609 km	
11	1 in	=	2.54 cm	
13	1 ft	=	12 in	
14	1 nm	=	10^{-9} m	(nanometers)
15	1 Å	=	10^{-10} m	(Angstroms)
16	1 ly	=	9.5×10^{12} km	(light years)
17	1 AU	=	1.5×10^{8} km	(astronomical units)
18	1 NM	=	1.1516 mi	(nautical miles)

Table 2. Equations for converting selected lengths. Conversion number coordinates with Table 1.

Generating Unit Conversions

You do not need to memorize a long list of conversion factors that might arise in solving geological-mathematical problems, because many can be calculated from a few. There are 18 unit conversions in Tables 1 and 2. Several of them are unnecessary, because they can be derived easily from the others. Several more are unnecessary if one knows the prefixes (e.g., *kilo*- for 1000; *centi*- for one-hundredth; *milli*- for one-thousandth).

One knows twice as many conversions by knowing how to invert equations. For example, conversion 5 (Tables 1 and 2) says that there are 3.281 feet in a meter. From this, conversion 6, that there are 0.3048 meters in a foot, follows immediately. The algebra is:

$$1 \text{ m} = 3.281 \text{ ft}; \qquad (10)$$

dividing both sides by 3.281 produces

$$0.3048 \text{ m} = 1 \text{ ft}. \qquad (11a)$$

Similarly, conversions 9 and 12 are unnecessary, because they are simply inverted forms of 8 and 11, respectively.

Conversions can be combined to generate new ones. This is done by nesting equations (successive substitution). For example, how many centimeters are there in a foot? According to Equation 11a,

$$1 \text{ ft} = (0.3048)(1 \text{ m}). \qquad (11b)$$

Substituting the 1 m of Equation 9 into the 1 m of Equation 11b produces

$$1 \text{ ft} = 30.48 \text{ cm}, \qquad (12)$$

which is conversion 7 of Tables 1 and 2. Therefore, that conversion does not need to be listed in the tables because it is implied by two conversions that are already included (1 and 5).

Even Equation 5 does not need to be in the list. This conversion follows from conversions 1, 11 and 13:

$$1 \text{ m} = 100 \text{ cm}, \qquad (13a)$$
$$1 \text{ in} = 2.54 \text{ cm}, \qquad (13b)$$
$$1 \text{ ft} = 12 \text{ in}. \qquad (13c)$$

Thus, inverting the second and third equations,

$$1 \text{ cm} = 1/2.54 \text{ in.}, \qquad (13d)$$
$$1 \text{ in} = 1/12 \text{ ft}. \qquad (13e)$$

Then, by substituting these one by one into Equation 13a,

$$1 \text{ m} = (100)(1/2.54)(1/12 \text{ ft}) = 3.28084 \text{ ft}. \qquad (13f)$$

[In case you are wondering about the number of significant digits, there are *exactly* 100 centimeters in a meter, *exactly* 12 inches in a foot, *exactly* 3 feet in a yard, all by definition, and *exactly* 3937/3600 yards in a meter by Act of U.S. Congress, 1866. From these, a meter contains 39.37 in. (exactly), a meter contains 3.2808333 ft (rounded), and an inch contains 2.540005 cm (rounded).]

As further illustration (and practice with scientific notation), consider these conversions that are not included in the list:

How many Angstroms are in a micron?

How many light years are there in an astronomical unit? (A light year is the distance that light travels in a year. An astronomical unit is the average distance of the Earth from the sun.)

For the first problem, combine equations of conversions 2, 3, and 15 of the tables. For the second problem, combine conversions 16 and 17. The answers are:

$$1 \text{ Å} = 10^{-4} \ \mu\text{m} \ , \tag{14}$$

and

$$1 \text{ ly} = 6.3\text{x}10^{4} \text{ AU.} \tag{15}$$

(Inverting Equation 15 produces 1 AU = 1.58×10^{-5} ly, which means that light travels from the sun to Earth in 1.58×10^{-5} years, or 8.3 minutes.)

From length conversions, one can easily derive area and volume conversions. Armed with just a few conversions, an understanding of the prefixes, and some algebra, you are ready for many problems. For example, the seven conversions in Table 3 would serve for all

1.	1 in = 2.54 cm
2.	1 ft = 12 in
3.	1 mi = 5280 ft
4.	1 ha =$(100 \text{ m})^2$
5.	1 mi^2 = 640 ac
6.	1 mL = 1 cm^3
7.	1 ft^3 = 7.481

Table 3. Seven independent conversion equations for lengths, areas, and volumes

hydrogeology problems that I can think of that do not bring in units of time, mass, force, or their combinations. (Notice, there are only three lengths; although veterans of hydro courses know the number of feet in a meter and the number of kilometers in a mile, those are not *independent* of the conversions that are listed.) For example, consider these:

How many gallons (gal) are there in an acre-foot (ac-ft)? (An acre-foot is sometimes used for large quantities of water. It comes up in the following way. Suppose a lake covers 100 acres, and the water rises two feet. The volume has changed 200 acre-feet, because one acre-foot is the volume of a slab one acre in area and one foot thick).

How many liters are in a gallon (gal)?

A square mile contains how many hectares (ha)? (A hectare is a metric unit.)

For the first problem, use variations of Equation 5 of Table 3 and Equation 3 of Table 3:

$$1 \text{ acre} = 1/640 \text{ mi}^2, \qquad (16a)$$
$$1 \text{ mi}^2 = (5280 \text{ ft})^2 \; . \qquad (16b)$$

Combine these two equations with

$$1 \text{ ac-ft} = (1 \text{ ac})(1 \text{ ft}) \, , \qquad (16c)$$

to get

$$\begin{aligned} 1 \text{ ac-ft} &= (1/640 \text{ mi2})(1\text{ft}) \\ &= (1/640)(5280 \text{ ft})^2(1 \text{ ft}) \\ &= \quad 43{,}560 \text{ ft}^3. \end{aligned} \qquad (16d)$$

Then combine this result with Equation 7 of Table 3,

$$\begin{aligned} 1 \text{ ac-ft} &= (43{,}560)(7.481 \text{ gal}) \\ &= \quad 325{,}900 \text{ gal}, \end{aligned} \qquad (16e)$$

to four significant figures (Equation 7 of Table 3is not exact).

For the second problem, combine Equations 1 and 2 of Table 3 for

$$1 \text{ ft} = 30.48 \text{ cm},$$

which you already know. Then cube both sides, replace the cm^3 with milliliters (mL, Equation 6 of Table 3), combine the result with 1000 mL = 1 L, and then substitute for the ft^3 using Equation 7 of Table 3. The result is

$$1 \text{ L} = 0.2642 \text{ gal}, \qquad (17)$$

to four digits. (You may have noticed that there are about 4 liters to the gallon at gas stations that are going metric; the conversion is 1 gal = 3.785 L.)

Finally for the hectare, you can use

$$1 \text{ m} = 3.281 \text{ ft} \, ,$$

which can be obtained from Equations 1 and 2 of Table 3, and 1 m = 100 cm, as we have discussed. From this and Equation 4 of Table 3, find the number of square feet in a hectare. From Equation 3 of Table 3, find the number of square feet in a square mile. This gives two equations involving square feet; combine them to find that

$$1 \text{ mi}^2 = 259.0 \text{ ha} \, , \qquad (18)$$

to four digits.

The Failsafe Way

In principle, there is nothing wrong with the method of successive substitution for converting units. In practice, however, it is easy to make a mistake. Also, it is tempting to skip steps and do some of the substitutions in your head. That invites problems.

Most people prefer the method of "multiplying by one" because it is easier to keep track of what you are doing. Further, it catches some mistakes.

The method starts with the same equations. For example, consider the acre-foot problem again (an acre-foot contains how many gallons?). The equations are

$$1 \text{ mi}^2 = 640 \text{ ac}, \quad (19a)$$
$$1 \text{ mi} = 5280 \text{ ft}, \quad (19b)$$
$$1 \text{ ft}^3 = 7.481 \text{ gal}. \quad (19c)$$

Now, instead of working with the equations in this form, rearrange each of them to equal one. Thus,

$$1 = (1 \text{ mi}^2)/(640 \text{ ac}) = (640 \text{ ac})/(1 \text{ mi}^2), \quad (19d)$$
$$1 = (1 \text{ mi})/(5280 \text{ ft}) = (5280 \text{ ft})/(1 \text{ mi}), \quad (19e)$$
$$1 = (1 \text{ ft}^3)/(7.481 \text{ ft}) = (7.481 \text{ ft})/(1 \text{ ft}^3). \quad (19f)$$

Notice there are two different ways of making "1" for each of the equations.

Then, the idea is to select the forms of "1" that cancel out units in such a way that the units you want to end up with are on the left side of the equation and the units you are starting with are on the right. In this example, you want to end up with acre-foot, which, again, is an acre times a foot:

$$1 \text{ ac-ft} = (1 \text{ ac})(1 \text{ ft}).$$

Cancel out the acres by multiplying it by the first ratio of Equation 19d:

$$1 \text{ ac-ft} = (1 \text{ ac})(1 \text{ ft})[(1 \text{ mi}^2)/(640 \text{ ac})].$$

which will leave units of mi^2-ft. To get rid of the mi^2, multiply it by the square of the second ratio of Equation 19e; this will produce ft^3. Then multiply the ft^3 by the second ratio of Equation 19f, and you are left with gallons. Usually this is all done in one line:

$$1 \text{ ac-ft} = (1 \text{ ac})(1 \text{ ft})(1\text{mi}^2/640\text{ac})(5280 \text{ ft}/1 \text{ mi})^2(7.481 \text{ gal}/1 \text{ ft}^3)$$
$$= 325{,}900 \text{ gal}.$$

Notice how the units cancel out to produce gallons on the right. This canceling of units is the failsafe part. If the units do not cancel out right, you have made a mistake, probably by selecting the wrong ratio (e.g., multiplying rather than dividing by the conversion factor).

For completeness, here are the lines that do the other example conversions.

$$1\text{ L} = (1\text{ L})(1000\text{ cm}^3/1\text{ L})(1\text{ in}/2.54\text{ cm})^3(1\text{ ft}/12\text{ in})^3(7.481\text{ gal}/1\text{ ft}^3)$$
$$= 0.2642\text{ gal.}$$

$$1\text{ mi}^2 = (1\text{ mi}^2)(5280\text{ ft/mi})^2(1\text{ m}/3.281\text{ ft})^2(1\text{ ha}/(100\text{m})^2)$$
$$= 259.0\text{ ha.}$$

One does have to be careful to put the exponent on the correct side of parentheses. Dividing by $(2.54\text{ cm})^3$, which is correct in the liters-to-gallons problem, is not at all the same thing as dividing by 2.54 cm^3.

The second example of rhetorical algebra, where plate speed was converted from cm/yr to km/m.y., is another example of the method of multiplying by one.

Using Unit Conversions

Geologists don't sit around converting units for fun. They convert units when there is a need for it. The need arises when they do geological-mathematical calculations, because the measured quantities are not always in the units needed to produce the desired result. Unit-conversion problems are embedded within more substantive problems.

The examples of CG-6 illustrate a common way that unit conversions arise. The spatial and temporal scales of geological problems are such that a person needs to go from meters to microns or kilometers, from grams to kilograms, and from years to millions of years with ease. The metric system, of course, is beautifully suited to such changes by powers of ten. Also, like it or not, it is common to have a problem where the given data are in a variety of English units (e.g., rainfall in inches/year, stream widths and depths in feet, stream lengths in miles, and drainage area in acres). And, problems with mixed units can arise because some new measuring devices are metric, whereas old data and maps with English units are still in use.

Another way that the need to convert units arises is worthy of special note, because its dangers tend to be underappreciated. Many equations, especially in geomorphology and watershed hydrology, are of the form

$$y = ax^b \,, \qquad (20)$$

where b is a weird, non-integer exponent, such as 0.237 or 3.58. Such equations result from fitting a straight line to a graph of log y vs. log x (as opposed to being derived from first principles). The snag is that these empirical equations are specific to the units for which the equation was found. (The appropriate language for these equations is that they are *not dimensionally homogeneous*, which is another story; suffice it to say that the units on one side of the equation are not consistent with those on the other, if the a is thought of as a unitless coefficient).

As an example, consider one of the pioneer equations in quantitative geomorphology:

$$L = 1.4A^{0.6} \,, \qquad (21)$$

from Hack (1957). Cited in nearly every geomorphology textbook, this equation relates the lengths of streams (L) to their drainage areas (A) in the Potomac River basin. A comparable equation has been found to apply at many other places around the world. One can use this equation to get an idea of the size of a drainage basin after measuring the length of the trunk stream. But, the L must be in miles, and the A must be in square miles (and many books don't say this).

To illustrate, consider this problem: What is the predicted stream length if the drainage area is 43 km^2? if you substitute the 43 km^2 into Equation 21 and calculate L, you get 13.4 km, *which is the wrong answer*. To get the right answer, you need to convert the 43 km^2 to 16.61 mi^2. Substituting that value for A produces $L = 7.56$ mi, which converts to 12.16 km (including excess digits).

Equation 21 is simple enough that the equation itself can be easily converted to use kilometers directly. To do this you must see that the "1.4" of Equation 21 is in units of $mi^{-0.2}$. So then, you need to convert 1.4 $mi^{-0.2}$ to units of $km^{-0.2}$, which you can do like any other unit conversion. Specifically, multiplying by one:

$$(1.4\ \text{mi}^{-0.2})(1.609\ \text{km/mi})^{-0.2} = 1.27\ \text{km}^{-0.2}$$

The kilometer version of Hack's Equation, therefore, would be

$$L = 1.27\, A^{0.6}\ ,$$

where L is in km and A is in km^2.

Empirical equations, which can be extremely useful, can get fairly complicated. For example, here is one from a recent textbook on watershed hydrology (McCuen, 1998, Eq. 7-6) (actually I am reducing the number of digits in the coefficient and exponents):

$$q = 5.44A^{0.680} S^{0.500} L^{0.168} P^{0.709} R^{0.150} D^{0.165} C^{1.26}$$

In this equation, the 10-year peak discharge (q, in ft^3/sec) at 58 gaged sites in Indiana is expressed in terms of drainage area (A, mi^2), channel slope (S, ft/mi), channel length (L, mi), a precipitation index (P, in.), basin relief (R, ft), drainage density (D, mi/mi^2), and a soil-runoff coefficient (C, no units). The usefulness of this equation is that it allows prediction of the 10-year peak discharge (important for flood design) at similar ungaged streams in the area. But it has to be used correctly. How would you like to go to the trouble of calculating a value for q after substituting in metric values, and then find out that your result is nonsense because you didn't use the right units?

Concluding Remarks

The notoriety of American students for being weak in algebra is an outcome of the release of the findings of TIMSS in early 1998. TIMSS is the Third International Mathematics and Science Study, which tested samples of fourth-, eighth- and twelfth- grade students from many countries around the world. The press release of February 24th, 1998, was followed the next day by newspaper articles with headlines such as this one in the NY Times (On the Web): "U.S. High School Seniors Among Worst in Math and Science". The article quoted President Clinton: "There is something wrong with the system and it is our generation's responsibility to fix it. You cannot blame the schoolchildren. There is no excuse for this." (There is a large volume of information about TIMSS now. Places to start are: http:/www.enc.org;

http://ustimss.msu.edu; http://nces.ed.gov/timss; http://www.col-ed.org; http://wwwcsteep.bc.edu/timss.)

TIMSS was the most comprehensive and rigorous international comparison of schooling ever undertaken. The U.S. was one of 21 countries in the comparison of twelfth-graders. Students from traditionally high-performing Asian countries were not included. In general knowledge of mathematics, American twelfth-graders did better than those of only Cyprus and South Africa. General knowledge was defined as the knowledge of mathematics needed to function effectively in society as adults. The general mathematics included fractions and percentages, graphs, and some algebra.

(Poor results on earlier studies had been dismissed on the notion that American scores were diluted by the wider range of students taking the exams. In TIMSS, this effect was controlled by giving a separate exam to students who had taken or were taking pre-calculus, calculus or advanced placement calculus. The U.S. came in twelfth of the twelve countries participating in this "advanced mathematics" assessment. Only when the students who took advanced placement calculus were isolated as a statistical group, did American students reach the international average.)

From the many discussions of TIMSS on the Internet, it is clear that people have drawn a variety of conclusions from the results. My own is influenced by one of the themes in Katz's review of algebra education. He made the point that, throughout history, algebra has been about solving problems, yet when algebra problems in textbooks have involved applications, they have seemed artificial and contrived. It makes one wonder, what is there to convince students that school algebra is useful?

Admitting that I am biased, I conclude that students would benefit from solving interesting geological-mathematical problems. Do them in geology-majors courses, in introductory geology courses, in science-education courses, in secondary schools. Embedded within these problems are many manipulations -- such as unit conversions -- that provide context and practice for the school algebra of manipulating symbols and equations to find unknowns.

Sources

Of particular help to me were the chapters "The Mathematics of Islam" and "Algebra in the Renaissance" in Katz (1998) and a variety of indexed items in Boyer and Merzbach (1991). As usual, I drew from overviews and capsules in NCTM (1989): "al-Khowarizmi" (D. Schrader, p. 76-77); "The history of algebra" (J.K. Baumgart, p. 233-260); "Equations and the ways they were written" (K. Cummins, p. 260-263).

References Cited

Boyer, C.B. and Merzbach, U.C., 1991, A history of mathematics, 2nd edition: New York, Wiley, 715 pp.

Hack, J.T., 1957, Studies of longitudinal stream profiles in Virginia and Maryland, U.S. Geological Survey Professional Paper, 294B, p. 45-97.

Katz, V.J., 1998, A history of mathematics, An introduction, 2nd edition: Reading, Addison Wesley, 864 pp.

Lacampagne, C.B., Blair, W. and Kaput, J., 1993, The Algebra Initiative Colloquium microform: papers presented at a conference on reform in algebra, December 9-12, 1993, 2 vols, Washington D.C.: U.S. Department of Education, Office of Educational Research and Improvement, National Institute on Student Achievement, Curriculum and Assessment.

McCuen, R.H., 1998, Hydrologic analysis and design, 2nd edition: Upper Saddle River, Prentice Hall, 814 pp.

NCTM (National Council of Teachers of Mathematics), 1989, Historical topics of the mathematics classroom: Reston, NCTM Inc, 542 pp.