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Computational Geology 6

Solving Problems

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Introduction

1. Figure 16.15 of the physical-geology textbook, *The Dynamic Earth* by B.J. Skinner and S.C. Porter (3rd ed., Wiley, New York, 1995, 567+ pp.), is a pair of maps of the Hawaiian-Emperor chain. Annotating the maps are numbers showing radiometric ages of basaltic rocks of shield volcanoes underlying those islands, atolls, and seamounts. According to the map showing the Hawaiian islands, the age at eastern Oahu (the island where Honolulu, Waikiki Beach, and Diamond Head are) is 2.6 million years (Ma), and the age at the northern tip of the island of Hawaii (the "Big Island") is 430,000 years. The map also includes a graphic scale showing distances of 50, 100 and 150 km, from which it appears the distance between the localities with the cited ages is about 250 km. The explanation for the pattern of ages is familiar to geology students: the Pacific Plate has been passing over a hotspot that produced a shield volcano at what is now eastern Oahu 2.6 Ma and a shield volcano at what is now northwestern Hawaii 0.43 Ma. So, given these data, how fast was the Pacific Plate moving from 2.6 to 0.43 Ma?

2. An echo sounder in an ocean vessel generates a pulse of sound waves that travel down to the seafloor, reflect from it, and return to the vessel, where the signal is received and the elapsed time is measured. If the round-trip transit takes 4.62 sec, how deep (m) is the water according to the device, if it is programmed to assume a value of 1490 m/sec for the speed of sound in seawater?

3. Interested in how long it takes for foraminifera to settle to the bottom of the ocean after they die, you make a collection of dead forams from ocean sediments. You clean them and separate out a sample of planktonic forams ranging from 125 to 175 microns (μm) in diameter. Then you set up a 110-cm settling tube in your lab, which is at 25º C temperature, fill the tube with deionized water, and let your sample of forams settle through it. You time their fall to the bottom as 185 seconds (an average). How long would it take an average foram to settle to a deep part of the abyssal plain, say, at a depth of 5.5 km (ignoring effects of dissolution)?

Word problems such as these illustrate how mathematics can be applied. This column is about geological-mathematical word problems.

On Word Problems

Like many students, you may believe that geological-mathematical word problems simply offer torment. To try to convince you to the contrary, let me tell you about the champion of problem solving, George Polya, the legendary mathematics educator. Polya's book, *How to Solve It* (2nd ed., Princeton University Press, 1957, 253 pp.) made the case for problem solving. In the preface of his book, Polya took the position that (*How to Solve It*, p. v)

> ... a teacher of mathematics has a great opportunity.... [If the teacher] challenges the curiosity of students by setting them problems proportionate to their knowledge, and helps them to solve their problems with stimulating questions, the teacher may give them a taste for, and some means of, independent thinking

Polya was interested in the "stimulating questions" that teachers would ask to help their students solve problems. According to him (*How to Solve It*, p. 1-2),

> ... the teacher is led to ask the same questions and to indicate the same steps again and again. Thus, in countless problems, we have to ask the question: *What is the unknown?* We may vary the words, and ask the same thing in many different ways.... Sometimes, we obtain the same effect ... with a suggestion: *Look at the unknown!* [Such questions and suggestions] aim at the same effect: they tend to provoke the same mental operation.

And so, Polya wrote about "*mental operations typically useful to the solution of problems*" (Polya's emphasis, p. 2). He called the study of methods of solution *heuristic* (p. vii), and claimed, "Heuristic aims at generality, at the study of procedures which are independent of the subject-matter and apply to all sorts of problems" (p. 133).

In *How to Solve It*, Polya classified the questions and suggestions that are helpful in discussing problem solving into four sequential categories. This classification constitutes "Polya's Four Steps", which are reproduced in varying forms in numerous mathematics-education texts. The four steps are: (1) understanding the problem, (2) devising a plan, (3) carrying out the plan, and (4) looking back.

Polya expanded his views in a two-volume work, *Mathematical Discovery: On Understanding Learning, and Teaching Problem Solving* (Wiley, v. 1, 1962, 216 pp; v. 2, 1965, 191 pp.). Following are three concepts that you might find interesting on the importance of problem solving in general, and word problems in particular. (All quotes are from v. 1.)

> **1. Know-how is the important thing**
>
> (p. vii-viii): Our knowledge about any subject consists of *information* and of *know-how*. If you have genuine *bona fide* experience of mathematical work on any level, elementary or advanced, there will be no doubt in your mind that, in mathematics, know-how is much more important than mere possession of information.
>
> What is know-how in mathematics? The ability to solve problems -- not merely routine problems but problems requiring some degree of independence, judgment, originality, creativity.

1. UNDERSTANDING THE PROBLEM ➢ What is the unknown? ➢ What are the data? ➢ Are there constraints on the answer? ➢ Draw a figure. Annotate it. Show the data and unknown. ➢ Are the data sufficient to determine the unknown? Or, are there intermediate unknowns? Are there obstacles between the knowns and unknowns? Show these on your figure. ➢ Are there extraneous data? ➢ Are the data internally consistent? Are their units consistent?
2. DEVISING A PLAN ➢ Does the combination of knowns and unknowns remind you of a problem you have solved before? ➢ What is the connection between the data and the unknown? How can you get through or around the obstacles to get to the target? ➢ What are the relevant equations? Do they need to be rearranged? In what sequence do you need them?
3. CARRY OUT THE PLAN ➢ Convert to a consistent set of units. ➢ Use the equations. ➢ Check each step.
4. LOOKING BACK ➢ Is the result reasonable? Can you test it? ➢ Do the units work out? ➢ How many significant figures? ➢ Can you get the result a different way? ➢ Now that you have worked through the problem to the end, does the whole thing make sense? Can you see it all at a single glance?

Table 1. Polya's Four Steps Applied to Geological-Mathematical Word Problems

2. Problems, by definition, contain obstacles.

(p. v.): Solving a problem means finding a way out of a difficulty, a way around an obstacle, attaining an aim which was not immediately attainable.

(p. 117): A problem is a 'great' problem if it is very difficult, it is just a 'little' problem if it is just a little difficult. Yet some degree of difficulty belongs to the very notion of a problem: where there is no difficulty, there is no problem."

3. Do many problems.

(p. v.): Solving problems is a practical art, like swimming, or skiing, or playing the piano: you can learn it only by imitation and practice.... (I)f you wish to learn swimming you have to go into the water, and if you wish to become a problem solver you have to solve problems.

Polya's Four Steps

Polya was a mathematics educator experienced with mathematics students solving mathematical word problems. Of interest to this column are geologists solving geological problems with mathematics, that is, geological-mathematical word problems. In the last column (CG-5, "If geology, then calculus), I applied Polya's four steps to what Polya would call "a problem to prove": prove the proposition, "If geology, then calculus." In this column, I will illustrate the use of Polya's heuristic to solve the geological-mathematical problems posed in the introduction. Polya would call these "problems to find."

Polya's four steps are elaborated in Table 1. Many of the bulleted items are taken from *How to Solve It*. I have added some others for their particular relevance to the kind of problems that geology students may encounter.

Polya's four steps were not the last word in problem solving by any means – not even for mathematical problems. A more-recent, influential author is Alan Schoenfeld, a mathematics-education researcher. In his book *Mathematical Problem Solving* (Academic Press, Orlando, 1985, 409 pp.), he described his research of videotaping pairs of math students attacking – more precisely, verbalizing their thoughts while attacking – mathematics word problems. He concluded that heuristic is only one of four categories of knowledge and behavior relevant to students' success in solving problems. The others are: (1) the students' background knowledge that they can bring to the attack; (2) "control" issues, or command-type decisions, such as choosing between alternative strategies and knowing when to backtrack, change direction, or start over; and (3) "belief systems", meaning the individual attitudes about self (e.g., one's abilities), the setting, the particular type of problem, and mathematics and problem solving in general.

The Hawaiian-Volcanoes Question

The target unknown is the velocity of the Pacific Plate. Plate motions are typically discussed in units of cm/yr. The data are: 2.6 m.y for one locality, 0.43 m.y. for another locality, and 250 km for the distance between the two localities. The unknown and data are shown in Figure 1.

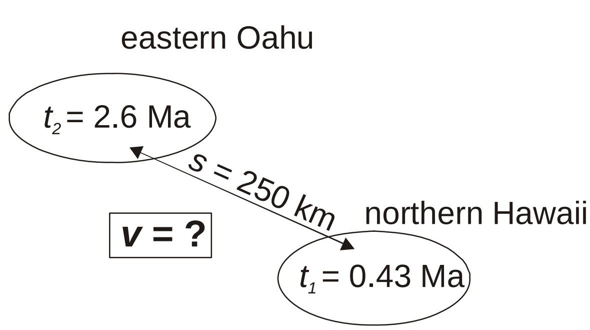

Figure 1. Sketch for the Hawaiian-volcanoes question asking for the speed of the Pacific plate fro age and distance data (Not to scale).

The question asks for a speed. The definition of speed is the distance the object in question travels divided by the time it takes for the object to travel that distance. In equation form, the definition is:

$$v = \frac{s}{\Delta t} = \frac{s}{t_2 - t_1}, \quad (1)$$

where v is speed, s is distance traveled from locality 1 to locality 2, Δt is the travel time, which in turn is the clock time (t_2) that the object arrives art locality 2 less the clock time (t_1) that the object was at locality 1. In this particular question, Δt is the difference in the ages. The data are sufficient to determine the speed. There are no extraneous data. The constraint on the answer is that the speed must be understood to be the average speed for the time interval, 2.6 to 0.43 m.y. ago.

The data are in units that are not consistent with those of the desired answer. In particular, the distance is in kilometers and the desired speed is in cm/yr. Recalling that there are 1000 meters in a kilometer, and 100 centimeters in a meter (*kilo*- meaning 1000, and *centi*- meaning 0.01), you have 1 km = 10^5 cm. Then, using scientific notation, the data in consistent units are: $s = 250\times10^5$ cm, $t_2 = 2.6\times10^6$ yr, and $t_1 = 0.43\times10^6$ yr.

Plugging the data into the equation, you have:

$$v = (250\times10^5 \text{ cm})/(2.6\times10^6 \text{ yr} - 0.43\times10^6 \text{ yr}). \quad (2)$$

Chugging through the math produces:

$$v = (25\times10^6 \text{ cm})/(2.17\times10^6 \text{ yr}) = 11.5207 \text{ cm/yr}. \quad (3)$$

The units clearly work out (cm over yr on one side of the equation, and cm/yr on the other). The "11.5207" is from your calculator and ignores the whole question of significant figures. With the distance, s, good to only two significant digits (250 km), the result can have only two digits (see CG-1, "Significant Figures!" May, 1988). Note that the extra digit is kept in the intermediate step after the subtraction $t_2 - t_1$. The answer you report from equation 3 should be 12 cm/yr.

Is the answer reasonable? Let's come back to that question.

The Water-Depth Question

The target unknown is the depth of the ocean, which is a distance. The data are the speed and an elapsed time. The elapsed time is the time for the sound waves to travel the distance twice. The known and unknown quantities are shown in Figure 2, with v, Δt, and D indicating the speed, elapsed time, and water depth, respectively.

Have you worked a word problem like this? Yes, you just did. In the preceding problem, you calculated the speed from distance and elapsed time. You are asked now to calculate distance given the speed and elapsed time. The appropriate equation, therefore, is a turned-around version of Equation 1:

$$D = \frac{v\Delta t}{2} \quad (4)$$

where the “2” is present because the sound wave travels down *and* back in the elapsed time (i.e., $s = 2D$).

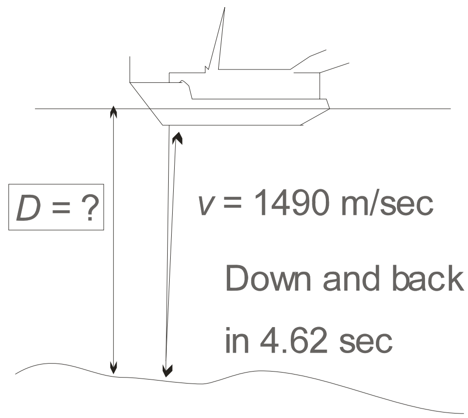

Figure 2. Sketch for the water-depth question asking for the depth to bottom from travel time and an average speed of sound. (Not to scale.)

The units are consistent, and there are no extraneous data. You can proceed directly to the plug and chug:

$$D = (1490 \text{ m/sec})(4.62 \text{ sec}) / 2 = 3441.9 \text{ m}$$

The units work out, m/sec times seconds producing meters.

You need to round off to three significant digits, because the Δt is known to only three figures. That gives you 3440 m for the water depth. You can report your answer as either 3.44×10^3 m or 34<u>4</u>0 m (i.e., underlining the second “4”) to avoid giving the impression that you know the depth to four significant digits (see CG-1).

The result is reasonable inasmuch as it is within the range of values in the deep ocean.

Those Questions Were Not Problems!

Those calculations did not have any obstacles. Remember, *problems, by definition, contain obstacles*. (Polya: “Where there is no difficulty, there is no problem.”) Therefore, those calculations were not problems. They were *exercises*. They involved a slight rearrangement of an equation and a straightforward conversion from kilometers to centimeters. Such manipulations do not qualify as “obstacles.”

Now, let’s make some problems out of those exercises.

The Hawaiian-Volcanoes Question, Part 2.

The age and distance data we used for eastern Oahu and northern Hawaii produced a speed of 12 cm/yr for the Pacific Plate. The issue of whether or not that speed is reasonable was left hanging. There are more data available in Figure 16.15 of *The Dynamic Earth*. In particular, the small-scale map includes a radiometric age of 27.2 Ma. for the shield volcano beneath Midway atoll. What does this Midway age imply for the speed of the Pacific Plate between 27.2 Ma and 0.43 Ma (the date from northern Hawaii)? Is it consistent with the Hawaii-to-Oahu result?

On the face of it, this question may seem no more difficult than the first one, and so it might not appear to rise to the level of "problem". But look at the data: between 27.2 Ma and 0.43 Ma, there were 26.77 million years. There is something missing. If you want speed, and you have elapsed time, you need the distance. I did not give you the distance.

If you have a copy of *The Dynamic Earth*, you might look at the map including both Midway and Hawaii expecting to use the scale. You won't see a scale. There's the obstacle!

The map showing Midway and Hawaii has ticks on the border showing degrees of latitude and longitude. This information is sufficient to calculate the distance between the location of the northern Hawaii age and the location of the Midway age. From these ticks, I get the latitude and longitude of the north tip of Hawaii to be 20° N and 156° W, respectively, and the latitude and longitude of Midway to be 27° N and 178° W, respectively.

So, calculating the plate-motion speed from the Hawaii and Midway data involves an intermediate step. The goal is to get the speed, and in order to get the speed you need to calculate the distance. The addition of an intermediate calculation to get from the given data to the target unknown qualifies as an "obstacle". To solve this problem now, you need an equation for, or a way of calculating, the distance between two points with known latitude and longitude. A way of calculating the distance from latitude and longitude data was given in CG-4, "Mapping with Vectors" (January 1999). Using that method, the great-circle distance between Hawaii and Midway is 2370 km (or 237×10^6 cm). Using this result for s and 26.77×10^6 yr for Δt in Equation 1 gives 8.853 cm/yr for the plate speed. With only two digits in the latitude, you need to round off to two digits in the answer (8.9 cm/yr).

To get back to the question of whether the first result is reasonable, I think one can conclude that the two results are consistent, given the uncertainties of reading off the map, the accuracy of the reproduction of the map itself, variations around the long-term mean of plate speed, and vagaries of sampling a shield volcano. Certainly it does not appear that we have made an error in going from the data that were provided to the target that was sought.

The Hawaiian-Volcanoes Question, Part 3.

This same problem and the same data can be recast in another way to make a point about scientific reasoning. Suppose you are given the data of the original question: the ages and Hawaii-Oahu distance. But suppose *you* are the person who has come up with the hotspot idea; it is *you* who *hypothesize* that the Hawaiian archipelago was formed by the motion of an ocean plate over a magma source. You can *quantify* your notion by the calculation we did in Equations 2 and 3. Thus your hypothesis is that the Hawaiian archipelago was formed by a plate moving at 12 cm/yr over a magma source. Then, with this *quantitative hypothesis*, you can make a *quantitative prediction*. You can predict the age of the volcanic rocks beneath Midway. This gives you a *test* of your hypothesis. The point to be made is that mathematics provides a way of deducing consequences from, and thus framing clear tests for, hypotheses. This is the well-known mathematico-deductive (or hypothetico-deductive) method of science. The person most identified with demonstrating the power of this role of mathematics is Galileo (1564-1642).

This entire line of thinking from data to hypothesis to prediction can be put as a geological-mathematic word problem as follows (see Fig. 3). Given the age and distance data for Hawaii and Oahu (i.e., 0.43 Ma. and 2.6 Ma, respectively, and a distance of 250 km between them), what is the age of the volcanic edifice beneath Midway, given the pertinent latitudes and longitudes?

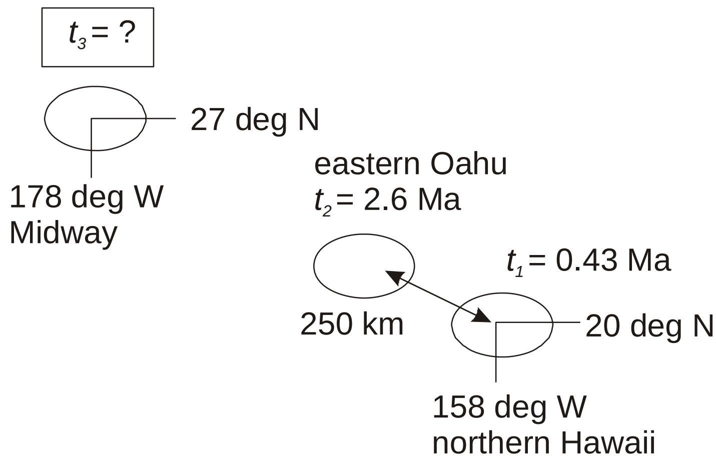

Figure 3. Sketch for the Hawaiian-volcanoes problem asking for a predicted age at Midway from ages and distance data for Hawaiian islands and the latitude and longitude of Hawaii and Midway. (Not to scale.)

Solving this problem requires several cognitive and mathematical steps: (1) recognizing that a speed is required as a go-between from the Hawaii-Oahu data to the Hawaii-Midway data; (2) calculating the plate speed from the "speed equation" (Equation 1) and Hawaii-Oahu data; (3) surmounting the obstacle of finding the Hawaii-to-Midway distance from the latitudes and longitudes; (4) using the "speed equation" again, but rearranging it for the Hawaii-Midway data; (5) calculating the elapsed time between Hawaii and Midway; and (6) calculating the predicted age at Midway. The math for these last three steps is:

$$\Delta t = \frac{s}{v} \tag{6}$$

$$\Delta t = (237 \times 10^6 \text{ cm}) / (11.52 \text{ cm/yr}) = 20.57 \times 10^6 \text{ yr} \tag{7}$$

and

$$t_2 = t_1 + \Delta t_2$$
$$t_2 = (0.43 \times 10^6 \text{ yr}) + (20.57 \times 10^6 \text{ yr})$$
$$t_2 = 21.0 \text{ Ma.} \tag{8}$$

So, in the end, you predict a value for Midway of 21.0 Ma from the Hawaii-Oahu data. Suppose you then drill Midway and find a radiometric age of 27.2 Ma. Would you consider that your test had failed, or passed?

The Water-Depth Question, Part 2.

Your answer to this last question – would you consider the test to have passed or failed? – would turn on (1) your assessment of the uncertainty of your calculated prediction given the uncertainties in the data going into the calculation, and (2) the flexibility of your hypothesis with respect to variations in plate-motion rates over time. Dealing with item (1) – uncertainties – can easily make the water-depth question into a problem.

Just as *The Dynamic Earth* is a standard textbook in introductory physical geology, *Oceanography, An Invitation to Marine Science* by T. Garrison (3rd ed., Wadsworth Publ. Co., Belmont CA, 1999, 540 pp.) is a standard textbook in introductory oceanography. The book states, “The speed of sound in seawater of 35‰ salinity is about 1,500 meters per second” (p. 172), but then the book goes on to include a figure showing that the speed of sound varies with depth. According to the graph of Figure 7.15 in the book, the speed of sound starts at about 1,500 m/sec at the ocean surface, decreases to about 1,480 m/sec at about 1,000-m depth, and then increases with depth, crossing 1,500 m/sec at about 3,000 m to end at the bottom of the graph at about 1,520 m/sec and 4,000 m. So here is the problem: *Allowing for this variation in the speed of sound*, what is the water depth if the echo sounder receives its signal back in 4.62 sec?

The equation is the same (Equation 4), and Δt is the same. Now the speed is an obstacle. From looking at the figure in the book, “about 1,500 m/sec” is clearly a good call, but there is considerable “slop” in the number. An average is called for. It cannot, however, be an average from the top to the bottom of the graph, because the depth -- the unknown -- may not be 4,000 m. What you need is an *average from the surface down to the bottom of the ocean at the location of the ship* -- but that depth, of course, is what you are trying to find! There is another obstacle. How do you get around it?

There are ways of solving this problem that would go beyond the immediate purpose of this column. But one easy thing to do is to give up on the notion of a precise answer. *Bracket your answer*. In other words, all possibilities could be covered by assuming a value of 1500 ± 20 m/sec for the speed of sound. While you are at it, why not figure in an uncertainty for the elapsed time? It was given as 4.62 sec. If the final “2” in “4.62” is significant, then the number really is 4.620 ± 0.005 sec (see CG-1, May, 1998). So now the question is: What is D from Equation 4, if $v = 1500 \pm 20$ m/sec and $\Delta t = 4.62 \pm 0.005$ sec?

The problem that has been framed now is a problem in error propagation: i.e., What error (uncertainty) is carried through to the end of a calculation from the errors (uncertainties) going into the calculation? A good source to consult about such problems is *An Introduction to Error Analysis* by John R. Taylor (2nd ed., University Science Books, Sausalito CA, 327 pp., 1997).

In that book, you find that the *maximum* uncertainty for a case like this is covered by the rule: “Relative errors add in multiplication and division.” A *relative error* is the absolute error divided by the number to which the absolute error is attached. For 1500 ± 20, the absolute error is 20, and the relative error is 20/1500 or 1.33%. Similarly, the relative error of Δt is 0.005/4.62 or 0.11%. The rule “relative errors add” means that the relative error propagated through Equation 4 to D is the sum of those two relative errors: 0.0144 or 1.4%.

You find D in the normal way. Plug 1500 m/sec and 4.62 into Equation 4, and you get 3465 m. With the relative uncertainty 0.144, the absolute uncertainty is 0.0144∗1500, or 49.9 m. Therefore, you have determined that the water depth is in the interval 3465 ± 50 m. You have bracketed it between 3415 and 3515 m.

Incidentally, you do not need to know the rule to get the same result. Use the minimum speed and minimum elapsed time to get the minimum depth, (1480)(4.615)/2, or 3415 m, and the maximum speed and maximum elapsed time to get the maximum depth, (1520)(4.625)/2 = 3515 m.

This result is a worst case with respect to the amount of uncertainty in a water depth determined from an elapsed time of 4.62 sec. There are ways of getting a much more precise answer. But as it is, this is a nice little problem. It is a little problem, because, quoting Polya, "it is just a little difficult."

The Sinking-Foram Problem

According to Polya, the threshold criterion for nonroutine problems is that they require "some degree of independence, judgment, originality, creativity." Nonroutine problems are those where "devising a plan" and *readjusting the plan* are major parts of the problem solution. These are more like problems that face geologists who apply mathematics. The third word problem in the introduction – the one about dead forams falling to the bottom of the ocean – has the makings of a nonroutine problem. We will look at it in some detail.

What is the target unknown? Answer: the time to reach the bottom of the ocean.

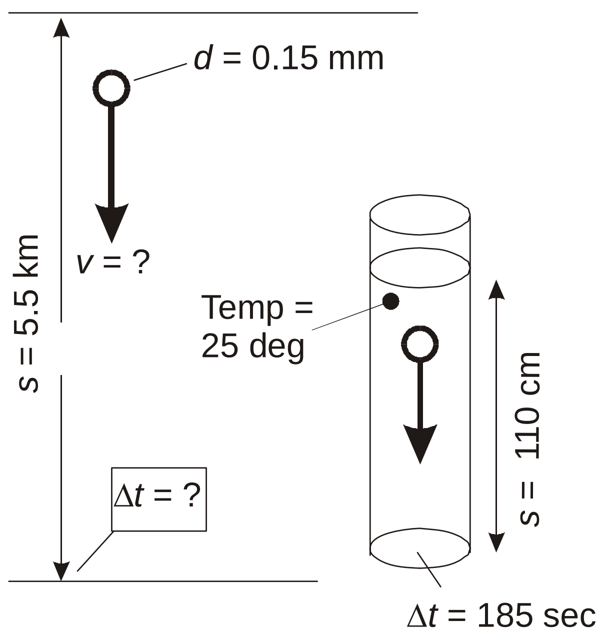

Figure 4. Sketch for the sinking-foram problem asking for travel time from travel distance and settling-tube data. (Not to scale.)

What information is given? Answer: the water depth, the size of the forams, and some laboratory data about those forams settling through a short column of water (see Fig. 4). Does the combination of data and unknown remind you of other problems? Yes: problems involving travel time, travel distance, and speed. Clearly you can calculate a settling velocity for the forams from the laboratory data using

$$v = \frac{s}{\Delta t} \; . \qquad (9)$$

But how does this equation help to find the unknown you were asked? Does this problem remind you of another problem?

This problem asks us to find the time to settle a long distance (the ocean) when we know the time to travel a short distance (the settling tube). In one of the Hawaii-volcanoes problems (Fig. 3), recall, we were asked to find a long travel time (Hawaii to Midway) from data about a short travel time (Hawaii to Oahu). In that problem, we went from the short time to long time by calculating the plate speed as an intermediate step. We can get an answer here by using the same strategy: calculate the settling velocity in the lab from Equation 9 and the lab data, and then use that velocity with the water depth to get travel time by way of

$$\Delta t = \frac{s}{v} \; . \qquad (10)$$

With that strategy in hand, carrying out the plan is simply a matter of working through Equations 9 and 10. Equation 9 becomes

$$v = (110 \text{ cm}) / (185 \text{ sec}) = 0.594 \text{ cm/sec} \; . \qquad (11)$$

To use Equation 10 with v in cm/sec, we need to change s = 5,500 m to s = 550,000 cm. Then Equation 10 becomes

$$\Delta t = (550{,}000 \text{ cm}) / (0.594 \text{ cm/sec})$$

$$= 925{,}926 \text{ sec} = 10.717 \text{ days} \; . \qquad (12)$$

So, using the laboratory-determined settling velocity directly produces an answer of 11 days (2 significant digits).

The answer, however, is wrong. Oh, the math is fine. As is often the case, more than math is required to solve this problem.

The Sinking-Foram Problem, Part 2

The preceding strategy assumes that the settling velocity of the forams in the lab is the same as it is in the ocean. Perhaps you recognized at the time that we blasted right through that assumption, and that it might get us into trouble. After all, the ocean is not deionized water. And its temperature is not 25^{o} C. Do we need to take those differences into account? If so, how? Now the question gets more interesting -- which is why it is a problem and not an exercise.

Schoenfeld's categories of knowledge and behavior that are relevant to problem solving are now coming into play. Recall, one such category is the students' background knowledge that can be brought to bear on the problem, and another is the students' decision-making abilities (i.e., judgment). By raising the question of what affects settling velocity, we enter the decision-making arena. Do we abandon the answer of 11 days? To address that question we have to know about settling velocity and factors that affect it. Background knowledge is crucial. What do you know about settling velocity?

You may recall that Stokes' Law speaks to settling velocity, and so you check indexes of your textbooks for Stokes' Law. If you do not recall or have never heard of Stokes' Law, then look up "settling velocity" in the indexes. Eventually you find Stokes' Law:

$$v = \frac{\rho_s - \rho_f}{18\mu} g d^2 , \qquad (13)$$

where v is the settling velocity of a small, spherical particle, ρ_s and ρ_f are the densities of the particle and fluid, respectively, μ is viscosity of the fluid, g is the acceleration of gravity, and d is the diameter of the spherical particle.

Equation 13 allows you to calculate the settling velocity ***if*** you know the vital statistics of the particles and the fluid. So you consult some references and come up with numbers for density and viscosity to represent the water column in the deep ocean: ρ_f = 1.03 g/cm^3, μ = 0.015 g/cm-sec. You know that foraminifera are calcite, and so you look in a mineralogy book to find the density of calcite as ρ_s = 2.71 g/cm^3. For the size of the particles, you assume your forams are spheres and take 150 μm for the diameter. In a physics book, you find g as 9.81 m/sec^2.

You have all the data for Equation 13, but the data are in a variety of units. This brings us to the subject of unit conversions, which we also encountered, to a lesser degree, in the earlier problems. The general rule is that you need to convert the quantities to a consistent set of units before you can plug them into an equation, or else you will calculate a nonsense result.

More mistakes seem to be made in unit conversions than in any other part of solving word problems. The subject warrants a whole "Computational Geology" column (next issue). For now, I will just note the conversions we need in this problem:

1. We need to decide on a single unit to use for all lengths. We will use centimeters, because density and viscosity, the data with the most complicated units, are given in units involving centimeters.

2. For g, 9.81 m/sec^2 converts to 981 cm/sec^2 because there are 100 cm in a meter.

3. For d, the 150 μm converts to 0.015 cm, because there are 1000 μm in a millimeter and 10 mm in a centimeter.

4. The densities and viscosity do not need to be changed.

We are done converting and can proceed with Equation 13

$$v = (2.71-1.03)(\text{g/cm}^3)(981\ \text{cm/sec}^2)(0.015\ \text{cm})^2 / (18(0.015\ \text{g/cm-sec})). \qquad (14)$$

Then we can proceed with Equation 10 and s = 550,000 cm:

$$\Delta t = (5.50\times10^5\ \text{cm}) / (1.3734\ \text{cm/sec})$$

$= 400{,}466$ sec = 4.635 days. (15)

Having carried out the plan, we go to Stage 4 of Polya's heuristic: Looking back. First, yes, the units work out, leaving us with cm/sec for the settling velocity and seconds (then days) for the time. Next, on the matter of significant figures, we clearly have carried excess digits through the calculation. We need to round off to two because of the viscosity, grain diameter and water depth. Our result, therefore, is 4.6 days for the forams to reach the bottom of the ocean.

But does 4.6 days for an answer seem reasonable?

No, it is not reasonable. Recall, *in the laboratory*, the forams had a settling velocity of 0.594 cm/sec. Now we calculate a larger settling velocity, 1.4 cm/sec, for when the forams are settling through *cold seawater*. This result is unreasonable in two ways. First, the effect of the greater salinity of seawater (greater ρ_f) is to increase the buoyancy of the particles because seawater is more dense than deionized water; therefore, the particles should settle more slowly. Second, the greater viscosity of the cold water means that there is a greater resistance offered by the fluid (viscosity measures the fluid resistance); therefore, again, the particles should settle more slowly. On both counts, therefore, the calculated settling velocity should be smaller than in the laboratory environment. Where did we go wrong?

The Sinking-Foram Problem, Part 3

We used Stokes' Law to predict the settling velocity of the forams in the ocean and, apparently, got an unreasonable result. Presumably, if we were applying Stokes' Law correctly, we would have gotten a reasonable result. It appears, therefore, that we may be applying Stokes' Law incorrectly. How can we test our use of Stokes' Law?

We can calculate a settling velocity and compare it directly to a known value. In the laboratory environment, the forams settle at 0.594 cm/sec. What do we get for the settling velocity in that environment, if we calculate it from Stokes' Law?

According to the *Handbook of Chemistry and Physics* (CRC Publ. Co., Cleveland, 1986) water at 25° C has a density of 0.997 g/cm^3 and a viscosity of 0.008937 g/cm-sec. With these new values, and with the old values of g = 981 cm/sec^2, d = 0.015 cm, and ρ_s = 2.71 g/cm^3, Equation 13 now gives:

$$v = (2.71\text{–}0.997)(\text{g/cm}^3)(981\ \text{cm/sec}^2)(0.015\ \text{cm})^2/\ (18(0.008937\ \text{g/cm-sec}))$$
$$= 2.3504\ \text{cm/sec} \qquad (16)$$

or 2.4 cm/sec. So, *the value we calculate for the settling velocity in the settling tube is four times larger than the value that was measured*. We must be doing something wrong with Stokes' Law. What is it?

There are at least three possibilities. First, the equation assumes that the particles are spheres, and our "spherical" forams may not be perfect spheres. Their actual shapes may offer more resistance than perfect spheres and, therefore, cause the forams to fall at a slower rate. Second, the combination of grain diameter, fluid density and fluid viscosity puts the Reynolds' number for both the measured and calculated settling velocities in the settling-tube experiment slightly outside the range where Stokes' Law is strictly valid. These are quibbles, however; a bigger effect is due to the third possibility.

The third explanation for the discrepancy between the calculated and measured settling-tube velocities is that we assumed that the *forams are solid calcite* when we plugged values into Stokes' Law. That is exactly what we said when we took the density of the particles (ρ_s) to be 2.71 g/cm^3, the density of calcite. Foraminifera, however, are *not* solid all the way through; they have chambers. The density of the forams must be much less than that of nonporous calcite. The smaller, *actual density* of the forams would make them settle more slowly than the solid calcite spheres that we assumed in our calculation.

This line of thinking means that we have been using an unrealistic value for ρ_s. How do we get a better one? We can use Stokes' Law again. Now the question is: What value of ρ_s in Equation 13 produces a value of 0.594 cm/sec for 150-μm-diameter spherical particles settling through water of density 0.997 g/cm^3 and viscosity of 0.008937 g/cm-sec? In other words, we need to work the Stokes' Law problem backwards. Without going into the rearrangement of the equation, let me just say that the answer is 1.43 g/cm^3. In other words, we can use the settling-tube data and Stokes' Law to "back out" (as they say) a density of the forams as 1.43 g/cm^3. Because we continue to assume that 150 μm is the real size of the forams and that they really are spheres, it is better to say that we have determined an *effective density*.

With this effective density, we can proceed finally to calculate the time for the forams to settle to the bottom of the ocean. Using ρ_s = 1.43 g/cm^3, ρ_f = 1.03 g/cm^3, μ = 0 .015 g/cm-sec, and d = 0.015 cm, Equation 13 and v become:

$$v = (1.43 – 1.03)(\text{g/cm}^3)(981\ \text{cm/sec}^2)(0.0015\ \text{cm})^2 / (18(0.015\text{g/cm-sec}) \\ = 0.327\ \text{cm/sec} \qquad (17)$$

Combining this result for v with s = 550,000 cm, Equation 10 and Δt become:

$$\Delta t = (550{,}000\ \text{cm}) / (0.327\ \text{cm/sec})\ 1.68\times10^6\ \text{sec} = 19\ \text{days} \qquad (18)$$

This answer combines the lab data, Stokes' Law, and the fluid properties of the ocean water. To review, we used Stokes' Law and data on settling velocity in a laboratory environment to back out an effective density of the spherical forams, and then used Stokes' Law to calculate their settling velocity and settling time in a different environment, the ocean.

We tried a variety of approaches and finally arrived at a result that seems to make sense in light of all the information given in the original question. (However, the question, as posed, does not consider everything that determines the travel time for a dead foram to reach the bottom of the ocean. For example, what about the spines, which would cause a greater resistance and slow the fall? How long do the spines survive? What about the protoplasm, which would also affect
the particle density? Both of these were factored out in both the settling tube experiment and the calculation.)

Final Comments

In addition to (1) background knowledge, (2) heuristic, and (3) decision-making judgment, the fourth ingredient of successful problem solving by students, according to

Schoenfeld, is their attitude and beliefs about problem solving. One of the beliefs of mathematics students, he says, is that a word problem, if it can be solved at all, can be solved in five to ten minutes given the right approach. In other words, if the student hasn't solved a particular problem with the chosen line of attack in that amount of time, then the student is inclined to abandon that approach and try another. This is an outcome, it seems, of exams: 5-10 problems in 50 minutes.

From my own experience, I can add that there seems to be a somewhat similar belief among many geology students who are asked to solve geological-mathematical word problems. The belief is that an experienced problem solver proceeds straight to an answer when faced with a geological-mathematical word problem. In other words, if a problem doesn't yield to the first line of attack, there is a tendency for these students to think, "Maybe this problem can't be solved", or "This problem requires more math than I know", or "I am not good at this", or "I hate these kinds of problems!" Such students do not seem to appreciate that fits and starts, trial and error, wrong turns, dead ends, and frustrated and exasperated language are simply par for the course in geological-mathematical problem solving, even (or especially!) among seasoned practitioners. I believe that such a lack of understanding of the kind of combat required to defeat realistic problems derives, again, from the unreal environment of the educational setting: exercises rather than problems, routine problems rather than nonroutine ones.

I also believe that if you wrestle with nonroutine problems, you will find them more interesting than routine exercises, precisely, and perhaps paradoxically, because these problems are more frustrating. Think of all the people who enjoy crossword and other kinds of puzzles in newspapers and game shows. People like puzzles. Real problems, which require trying this, and then trying that, are puzzles. What is the attraction of puzzles? Certainly, the engagement of the mind is part of it. But, I believe, the big attraction is the sheer satisfaction of solving them. The more challenging -- and the more frustrating -- they are, the more is the satisfaction when they are solved.

Try working on a problem that it takes a few days just to understand what the problem is, what is being asked. Then suppose it takes a few days and a few trips to the library to figure out how to approach it, and many hours to carry out the plan, to do the calculations. Then suppose you find your actions have produced a garbage result. But suppose that by now the problem has really engaged you; you can't let it go; it torments you. So you try something else, and something else, and something else. Eventually you get an answer, and that answer makes sense and survives all of your tests of it.

The satisfaction from solving such problems is the kind that sustains professional geologists as they puzzle through *research questions*, which can last for years. To many, the attraction of research is precisely the challenge that is provided by problems requiring judgment, originality, creativity, and perseverance to solve. Knotty – even better, tormenting – geological-mathematical word problems offer you a small taste of this challenge, frustration, and satisfaction.

Sources and Suggested Readings

Settling velocities of foraminiferal tests are discussed in detail by Berger and Piper (1972), Fok-Pun and Komar (1983), and Takahashi and Bé (1984). Berger and Piper (1972) is the source of the settling tube numbers used here. According to those authors, a value of 1.5

g/cm^3 for the density of a water-filled foraminiferal test (i.e., excess density, $\Delta\rho$, is 0.5 g/cm^3) corresponds to a 10-μm wall thickness in a foram with a diameter of 250 μm. According to Takahashi and Bé (1984), sinking speeds are better determined on individuals from net-collected plankton tows ($\Delta\rho$ = 0.021 to 0.603 g/cm^3) than on individuals from sediments ($\Delta\rho$ = 0.087-1.318 g/cm^3), because of the presence of spines on the former and encrusted shells of the latter. For a discussion of Stokes' Law and excess density in the context of the hydromechanics of living plankton, see the classic by Hutchinson (1967, Chap. 20).

The paper by Berger and Piper (1972) was an important step in arguing that the bulk of foraminiferal dissolution occurs on the seafloor rather than in transit (see Adelseck and Berger, 1975). For more on the settling behavior of biogenic particles – including modifications of Stokes' Law for nonspherical geometries and quantitative considerations of the effects of grain dissolution on settling velocity – see Lerman (1979), which is the source of the values used here for seawater density and viscosity. For the larger context of which sinking, dissolving foraminifera are a part, see Chapter 5 of Morse and Mackenzie (1990) on the carbonate geochemistry of pelagic sediments. According to Mackenzie (pers. comm., 1999), quantifying the relative magnitudes of the dissolution flux from fallen forams vs. the dissolution flux from falling forams is a topic of ongoing research.

On dismissing the two "less important" explanations for the four-fold discrepancy between calculated and measured settling velocities in the discussion of the settling-tube experiment, there are two classic references. Rubey (1933) showed that measured settling velocities differ little from theoretical Stokes' Law velocities for grain diameters up to about 0.14 m. Waddell (1934) showed that settling velocities for disks, which are closer to the shape of natural sedimentary particles, are only about 35% smaller than the spherical-particle Stokes' Law velocities.

For more-general discussions of Stokes' Law see Blatt et al. (1980), Hsu (1989), and Middleton and Wilcox (1994). The original paper is Stokes' (1851), a date which places Stokes' Law in the time of Lyell's *Principles of Geology* (1830) and Darwin's *On the Origin of Species* (1859).

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