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Computational Geology 4

Mapping with Vectors

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Introduction

Vectors – Who needs them? If you have had a course in physics, you know that physicists can't live without them. The reason is that key quantities such as velocity and force have direction, and vectors deal explicitly with direction.

But geologists, too, need to think about direction – all geologists, not only geodynamicists who deal explicitly with forces and enjoy physics. For geologists, no matter what else they may think about, must think about *location*, and thinking about location in a quantitative way quickly brings a person to *direction*.

As an example, consider your present location. Now think about some other location, say the location of your book that tells about maps (it might come in handy soon). What is the location of that book relative to you? Let's say that it is 2,730 m in a direction NNE.

Now, think of yourself as the origin of a planar coordinate system (on a planar world); then, the position of the book can be stipulated by two numbers, r and θ. The r is the distance, 2,730 m. The θ is an angle defining the direction and requires a little explanation. In order for θ to mean anything it must be measured from a reference line that is understood by everyone. This agreed-upon direction is the direction of positive x, which, for map work, is taken as due east. By convention, θ is measured counterclockwise from the x-axis. So when $\theta = 90°$, the point stipulated by r and θ lies on the y-axis in the direction of positive y, which means it lies due north of the origin (the x- and y-axes form a right-handed, Cartesian coordinate system). The r,θ coordinates are probably familiar to you; they are the *polar coordinates* of the point in question.

The polar coordinates of your book, then, are: r = 2,730 m and $\theta = 67.5°$ (recall, you are the origin of the coordinate system). If you are uncertain where the value of θ came from, get that book on maps and look up *bearing* and *azimuth*. The bearing of NNE is N22. 5° E, and the Azimuth is 22.5°. Then, θ is 90.0°–22.5° E or 67.5°.

Now, back to vectors, which have both magnitude and direction. The position of the book relative to you (or, in general, the origin of the coordinate system), can be thought of as a vector: the magnitude is 2,730 m, and the direction is $\theta = 67.5°$ counterclockwise from the x-axis. This vector is known as the *position vector*, which can be thought of as a directed distance from the center of a coordinate system to a point under examination. Think of it as an arrow starting at the origin of the coordinate system and ending at the point.

Vectors are generally written in a way that distinguishes them from mere directionless numbers (scalars). Sometimes the vectors are written with an arrow over the letter; sometimes they are represented in bold font. We will use bold letters here.

The position vector is denoted by **r**. This essay is about **r** and how it can help you deal with positions on the surface of the Earth – that is, geographic location as represented on maps.

Components of r.

Thinking of vectors in terms of magnitude and direction is intuitive for geologists. For purposes of calculation, however, it is generally more useful to think of them in terms of components along the axes of a Cartesian coordinate system: *x*,*y* for two-dimensional problems.

The vector describing a distance of 2,730 m in a direction 67.5° north of east can be thought of as the *vector sum* of two vectors, one along the *x*-axis and the other along the *y*-axis; specifically, the first is a vector with magnitude 1,045 m in the direction of due east, and the second is a vector with magnitude 2,522 m in the direction of due north (assuming the original data are good to four significant figures). Saying that they are combined by a vector sum means that the tail of one is placed at the origin of the coordinate system, and the tail of the second is placed at the arrowhead of the first; then the original position vector, **r**, is the vector that extends from the origin of the coordinate system to the arrowhead of the second (along the hypotenuse of the right triangle defined by the west-to-east and the south-to-north vectors). You can think of this vector sum as the result of walking 1,045 m to the east and then 2,522 m to the north; you wind up at a distance of 2,730 m in a direction of θ = 67.5° from where you started.

Note that you get to the same place by walking 2,522 m north first and then 1,045 m east. In other words, the order of terms in the vector sum does not matter. In the language of mathematics, vector addition is *commutative*.

This vector sum that produces **r** can be succinctly written as:

$$\mathbf{r} = 1045\,\mathbf{i} + 2522\,\mathbf{j} \tag{1}$$

where the numbers are in meters, and **i** and **j** are, themselves, vectors. The **i** and **j** are unit vectors, meaning that the magnitude of each of them is exactly 1, and they are dimensionless; all they do is indicate direction. The unit vector **i** extends in the positive *x*-direction, and **j** extends in the positive *y*-direction. The first term, 1,045**i**, is a vector extending 1,045 m in the direction of due east; the second term, 2,522**j**, is a vector of 2,522 m in the direction of due north.

Because *x* and *y* (and **i** and **j**) are at right angles to each other, the vector sum expressed in Equation 1 can be thought of as a right triangle with the west-east leg being 1,045 m, and the south-north leg being 2,522 m. So, by the Pythagorean Theorem, the magnitude of **r** in meters – or the length of the hypotenuse – is:

$$|\,\mathbf{r}\,| = \sqrt{1045^2 + 2252^2}\ , \tag{2}$$

where the vertical lines on the sides of **r** (i.e., the absolute value of **r**) denote the magnitude of the vector. Also (remembering your trig), the angle counterclockwise from east is:

$$\theta = \arctan\,(2522/1045) = \arctan\,(2.413) = 67.5° \tag{3}$$

Where did the numbers 1,045 and 2,522 in Equations 1 and 2 come from? The first is |**r**|cos(67.5°), and the second is |**r**|sin(67.5°).

You should note that converting between the two forms of the position vector – the distance-and-direction form, and the *xy*-components form – is the same as converting between polar and Cartesian coordinates of the point at the end of the position vector. For, if you are given the polar coordinates (r,θ) of a point relative to an origin, then the Cartesian coordinates of the point relative to the same origin are

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta\,. \tag{4}$$

This is the same as saying that to walk from the origin to the point (r,θ), you can walk a distance r*cos(θ) in the x-direction and then walk a distance r*sin(θ) in the y-direction. The vector **r**, then, can be expressed by:

$$\mathbf{r} = |\,\mathbf{r}\,|\cos(\theta)\,\mathbf{i} + |\,\mathbf{r}\,|\sin(\theta)\;\mathbf{j} \tag{5}$$

To transform the location of a point from Cartesian to polar coordinates, you use

$$r = \sqrt{x^2 + y^2} \quad \text{and} \qquad \theta = \arctan(y\,/\,x)\,. \tag{6}$$

These quantities give the magnitude and direction, respectively, of the position vector from the origin of the coordinate system to the point in question. Equations 2 and 3 are simply applications of Equations 6.

Finding yourself

One of the most basic problems in field geology is finding your location on a map; there's no point in making a serious observation if you can't record where it is. If you are on land, you generally can be guided by the contours on your map to figure out where you are. If it is totally hopeless, then – the old-fashioned way – you can sight on two landmarks that are on the map and triangulate your location. (The new-fashioned way is to use your GPS system, which, for the purpose of this column, we will say has dead batteries). If you are on the sea and close to shore, you can triangulate on two lighthouses to find the location of your boat before it locates a reef.

The routine way of solving this triangulation problem is graphically, as follows. First, find the two landmarks on the map: let's call them *A* and *B*. Second, in the field, determine the direction from you to each of the landmarks (with a compass); let's say the direction to *A* is N20E (or Azimuth 20°), and the direction to *B* is S60E (or Azimuth 120°). Third, on the map, draw a line through *A* with a direction of N20E, and a line through *B* with a direction of S60E. Where the two lines cross is your location on the map.

Now here is another way you can solve the problem. Think of your unknown location as the origin of a coordinate system, with the positive x-axis going due east and the positive y-axis going due north. The vector $\mathbf{r_A}$ gives the position of *A*, and the vector $\mathbf{r_B}$ gives the position of *B*. We do not know the length of these vectors, but we do know their direction.

Because we do not know the length of $\mathbf{r_A}$ and $\mathbf{r_B}$, let's forget about these vectors and their length for the moment and focus on unit vectors, $\mathbf{e_A}$ and $\mathbf{e_B}$, corresponding to $\mathbf{r_A}$ and $\mathbf{r_B}$, respectively. A unit vector, as you know, is a vector with a magnitude of 1; you can think of it as a vector that is of interest only because of its direction. The unit vectors **i** and **j** are the best-known examples, but there are others. Whereas **i** and **j** necessarily point in the direction of coordinate axes, the unit vectors $\mathbf{e_A}$ and $\mathbf{e_B}$ point in other specific directions: parallel to $\mathbf{r_A}$ and $\mathbf{r_B}$, respectively. So $\mathbf{r_A}$ can be written as $r_A\mathbf{e_A}$, where r_A (note, *not* bold) denotes the magnitude of

$\mathbf{r_A}$, and $\mathbf{e_A}$ denotes the direction of $\mathbf{r_A}$. Similarly, $\mathbf{r_B}$ can be written as $r_B\mathbf{e_B}$.

Now resolve $\mathbf{e_A}$ and $\mathbf{e_B}$ into their component vectors. Because $\mathbf{e_A}$ has the same direction as $\mathbf{r_A}$, the direction of $\mathbf{e_A}$ is N20E. The counterclockwise angle between the *x*-axis and $\mathbf{e_A}$, therefore, is $\theta = 70°$. Because $|\mathbf{e_A}| = 1$ ($\mathbf{e_A}$ is a unit vector), then:

$$\mathbf{e_A} = \cos(70°)\,\mathbf{i} + \sin(70°)\,\mathbf{j} = 0.3420\,\mathbf{i} + 0.9397\,\mathbf{j}, \tag{7}$$

again assuming that the measured directions are good to four significant figures. Similarly, from the direction of $\mathbf{e_B}$ (S60E), the counterclockwise angle between the *x*-axis and $\mathbf{e_B}$ is –30°, and so

$$\mathbf{e_B} = \cos(-30°)\,\mathbf{i} + \sin(-30°)\,\mathbf{j} = 0.8660\,\mathbf{i} - 0.5000\,\mathbf{j}, \tag{8}$$

Having established vectors giving the direction to the landmarks, you now bring in information about the location of the landmarks on your map. Given the map location of the landmarks and the measured directions to them in the field, you can determine your location on your map by use of similar triangles. Suppose, for example, the coordinates of Landmark *A* are (x_A, y_A), relative to the origin of the map's coordinate system; these map coordinates can be as simple as the number of centimeters east and north, respectively, from the lower-left (southwest) corner of the map sheet. Then letting (x,y) be the map coordinates of your location – relative to the same origin, the lower-left corner of the map – the distance that Landmark A is east of you on the map is $x_A - x$, and the distance that Landmark *A* is north of you on the map is $y_A - y$. The ratio of these two distances is the same as the ratio of the corresponding components of $\mathbf{e_A}$:

$$\frac{y_A - y}{x_A - x} = \frac{0.9397}{0.3420} = 2.748, \tag{9}$$

where the numbers 0.9397 and 0.3420 come from Equation 7. Similarly, letting (x_B, y_B) be the map coordinates of Landmark *B*,

$$\frac{y_B - y}{x_B - x} = \frac{-0.5000}{0.8660} = -0.5774, \tag{10}$$

from Equation 8. Equations 9 and 10 are two equations in the two unknowns, *x* and *y*. They are your map coordinates. Solve Equations 9 and 10 for *x* and *y*, and you have found where you are on the map.

Without intending to go far into the subject of solving simultaneous equations, I would like to point out that there is a very easy way of proceeding to an answer. First, get Equations 9 and 10 into the following form:

$$\begin{aligned} a_1 x + b_1 y &= c_1 \\ a_2 x + b_2 y &= c_2 \end{aligned}. \tag{11}$$

Then, the solution is given by:

$$x = \frac{c_1 b_2 - c_2 b_1}{a_1 b_2 - a_2 b_1} \quad \text{and}$$

$$y = \frac{a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1} . \tag{12}$$

Equations 12 are an application of Cramer's Rule, which you may have encountered in an algebra course in connection with determinants. The technique is named for Gabriel Cramer (1704-1752), a contemporary and countryman of the Swiss mathematician, Leonhard Euler (see the "Computational Geology" columns of the preceding two issues).

To illustrate how easy the computation is, let's complete the problem by assuming that Landmark *A* is located at $x_A = 25.5$ cm and $y_A = 42.3$ cm from the lower left corner of the map, and that Landmark *B* is located at $x_B = 32.3$ cm and $y_B = 9.6$ cm from the lower left corner. From these coordinates, and carrying four digits, Equations 9 and 10 in the form of Equations 11 are:

$$\begin{aligned} 2.748x - y &= 27.77 \\ -0.577x - y &= -28.24 \end{aligned} \tag{13}$$

Plugging the coefficients in Equations 13 into Equations 12, you have:

$$\begin{aligned} x &= \frac{(27.77)(-1) - (-28.24)(-1)}{(2.748)(-1) - (-0.5774)(-1)} \\ y &= \frac{(2.748)(-28.24) - (-0.5774)(27.77)}{(2.748)(-1) - (-0.5774)(-1)} \end{aligned} \tag{14}$$

Chug out the arithmetic of Equations 14, and you get $x = 16.8$ and $y = 18.5$, which means that your location on the map is 16.8 cm to the right and 18.5 cm up from the lower left corner of the map sheet.

Calculating distances

Now that you know where you are on your map, you can easily calculate your distance to each of the two landmarks.

Your map coordinates are (16.8,18.5). The map coordinates of Landmark *A* are (25.5, 42.3), and the map coordinates of Landmark B are (32.3, 9.6). All these coordinates are in centimeters on the map.

In general, the distance between two points (x_1,y_1) and (x_2,y_2) is given by the Pythagorean sum:

$$\text{distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \ . \tag{15}$$

This is the general form of the relation that we used way back at Equation 2. For the cases here (and with an extra digit),

$$| \mathbf{r_A} | = \sqrt{(25.5-16.8)^2 + (42.3-18.5)^2} = 25.34 \quad \text{and}$$
$$| \mathbf{r_B} | = \sqrt{(32.3-16.8)^2 + (9.6-18.5)^2} = 17.87 \qquad (16)$$

in centimeters on the map.

These distances on the map can be easily converted to distances in the field by consideration of the scale of the map. Let's say that the scale of the map is 1:100,000. Then every centimeter on the map corresponds to 100,000 cm in the field. Converting the centimeters to kilometers,

$$10^5 \text{ cm} = (10^5 \text{ cm})*(10^{-2} \text{ m/cm})*(10^{-3} \text{ km/m}) = 1 \text{ km}, \qquad (17)$$

you see that your distances in the field to Locations *A* and *B* are 25.3 km and 17.9 km, respectively. (Notice, too, how convenient metric units are.)

These distances are the magnitudes r_A and r_B, respectively, of position vectors $\mathbf{r_A}$ and $\mathbf{r_B}$ from you to Locations A and B. We also have the directions of $\mathbf{r_A}$ and $\mathbf{r_B}$ from the unit vectors $\mathbf{e_A}$ and $\mathbf{e_B}$ (Equations 7 and 8). Recalling that a vector is its magnitude multiplied by its unit vector, you have:

$$\mathbf{r_A} = \mathbf{r_A e_A} = 25.34* (0.3420\, \mathbf{i} + 0.9397\, \mathbf{j}) \quad \text{and}$$
$$\mathbf{r_B} = \mathbf{r_B e_B} = 17.87* (0.8660\, \mathbf{i} - 0.5000\, \mathbf{j})\ . \qquad (18)$$

Complete the multiplication, and you get the position vectors from you to *A* and to *B*:

$$\mathbf{r_A} = 8.7\, \mathbf{i} + 23.8\, \mathbf{j} \quad \text{and}$$
$$\mathbf{r_B} = 15.5\, \mathbf{i} - 8.9\, \mathbf{j}\ , \qquad (19)$$

where the numbers are in kilometers.

The same result can be obtained a more direct way. You have the coordinates of your location and the two landmarks in centimeters on the map. Convert these to coordinates on the ground in kilometers by using the map scale. This gives (16.8, 18.5) for your location, (25.5, 42.3) for Location A, and (32.3, 9.6) for Location *B*, all in kilometers. Then you can use a general equation for the position vector *from* Point 1 at (x_1, y_1) **to** Point 2 at (x_2, y_2):

$$\mathbf{r} = (x_2 - x_1)\, \mathbf{i} + (y_2 - y_1)\, \mathbf{j}\ . \qquad (20)$$

For the position vector from you to the two locations, Equation 20 gives

$$\mathbf{r_A} = (25.5 - 16.8)\, \mathbf{i} + (42.3 - 18.5)\, \mathbf{j} = 8.7\, \mathbf{i} + 23.8\, \mathbf{j} \quad \text{and}$$
$$\mathbf{r_B} = (32.3 - 16.8)\, \mathbf{i} + (9.6 - 18.5)\, \mathbf{j} = 15.5\, \mathbf{i} - 8.9\, \mathbf{j}, \qquad (21)$$

which is the same result as Equations 19.

Some other calculations

Before proceeding with a three-dimensional problem, it is useful to discuss some vector operations in the context of our two-dimensional position vectors of Equations 19.

We have already noted that vectors can be added. In Equation 1, for example, the terms stating the vector components of **r** are added together vectorally. Similarly, if we were to add $\mathbf{r_A}$ and $\mathbf{r_B}$ together we would get a vector with the length and direction that would take you *directly* from your position at (16.8, 18.5) to a point that you could also reach by walking N20E for 25.34 km and then S60E for 17.87 km. This vector is obtained by adding the *x*-components together and the *y*-components together:

$$\begin{aligned}\mathbf{r_1} + \mathbf{r_2} &= (x_1\,\mathbf{i} + y_1\,\mathbf{j}) + (x_2\,\mathbf{i} + y_2\,\mathbf{j}) \\ &= (x_1 + x_2)\,\mathbf{i} + (y_1 + y_2)\,\mathbf{j}.\end{aligned} \tag{22}$$

Thus (from Equation 19):

$$\begin{aligned}\mathbf{r_A} + \mathbf{r_B} &= (8.7\,\mathbf{i} + 23.8\,\mathbf{j}) + (15.5\,\mathbf{i} - 8.9\,\mathbf{j}) \\ &= 24.2\,\mathbf{i} + 15.0\,\mathbf{j}.\end{aligned} \tag{23}$$

The length and direction of this vector can be found from the Pythagorean sum and the arctangent, respectively:

$$|\,\mathbf{r_A} + \mathbf{r_B}\,| = \sqrt{24.2^2 + 14.9^2} = 28.4 \text{ km}$$

$$\theta = \arctan\left(\frac{14.9}{24.2}\right) = 32°. \tag{24}$$

Then, to find where this would get you, simply add the vector (Equation 23) to the coordinates of your original location:

$$(16.8, 18.5) + (24.2, 14.9) = (41.0, 33.4). \tag{25}$$

Similarly, two vectors can be subtracted:

$$\begin{aligned}\mathbf{r_2} - \mathbf{r_1} &= (x_2\,\mathbf{i} + y_2\,\mathbf{j}) - (x_1\,\mathbf{i} + y_1\,\mathbf{j}) \\ &= (x_2 - x_1)\,\mathbf{i} + (y_2 - y_1)\,\mathbf{j}.\end{aligned} \tag{26}$$

This results in a vector $\mathbf{r_2} - \mathbf{r_1}$ that extends *from* the arrowhead of $\mathbf{r_1}$ *to* the arrowhead of $\mathbf{r_2}$. In the case of our position vectors, $\mathbf{r_A}$ and $\mathbf{r_B}$, the vector $\mathbf{r_B} - \mathbf{r_A}$ extends from Landmark *A* at (25.5, 42.3) to Landmark *B* at (32.3, 9.6). From Equation 26, this vector is:

$$\begin{aligned}\mathbf{r_B} - \mathbf{r_A} &= (15.5\,\mathbf{i} - 8.9\,\mathbf{j}) - (8.7\,\mathbf{i} - 23.8\,\mathbf{j}) \\ &= 6.8\,\mathbf{i} - 32.7\,\mathbf{j}.\end{aligned} \tag{27}$$

This vector can also be obtained from the coordinates of the two landmarks:

$$\begin{aligned}\mathbf{r_B} - \mathbf{r_A} &= (32.3 - 25.5)\,\mathbf{i} + (9.6 - 42.3)\,\mathbf{j} \\ &= 6.8\,\mathbf{i} - 32.7\,\mathbf{j}.\end{aligned} \tag{28}$$

No matter how the vector is obtained, its length and direction is easily found from the Pythagorean sum and arctangent, respectively:

$$| \mathbf{r_B} - \mathbf{r_A} | = \sqrt{6.8^2 + (-32.7)^2} = 33.4 \text{ km},$$

and $$\theta = \arctan\left(\frac{-32.8}{6.8}\right) = -78°, \quad (29)$$

where θ indicates a direction of 78° south of east (Azimuth 168°, or S12E). Thus if you start at (16.8, 18.5), walk 25.3 km in a direction of N20E to Landmark *A*, and then turn and walk 33.4 km in a direction S12E, you will arrive at Landmark *B* (32.3, 9.6), which you could have walked to in 17.9 km in a direction of S60E. In other words:

$$\mathbf{r_A} + (\mathbf{r_B} - \mathbf{r_A}) = \mathbf{r_B}. \quad (30)$$

There are two ways of multiplying vectors together: the dot product and the cross-product. We will not need the cross-product here (see CG-14), but the dot product is very useful now because it can supply the angle between vectors.

The dot product of two vectors is defined as the product of their magnitudes times the cosine of the angle between them:

$$\mathbf{r_1} \bullet \mathbf{r_2} = | \mathbf{r_1} | \, | \mathbf{r_2} | \cos(\lambda), \quad (31)$$

where λ is the angle between $\mathbf{r_1}$ and $\mathbf{r_2}$. The dot product results in a scalar, and so it is also called the scalar product. From Equation 31, it can be shown (we won't do it) that the dot product is numerically the same as the sum of the product of the two *x*-components and the product of the two *y*-components. That is:

$$\mathbf{r_1} \bullet \mathbf{r_2} = (x_1\, \mathbf{i} + y_1\, \mathbf{j}) \bullet (x_2\, \mathbf{i} + y_2\, \mathbf{j}) = x_1\, y_1 + x_2 y_2. \quad (32)$$

Combining Equations 31 and 32 gives you a means of calculating the angle:

$$\cos(\lambda) = (x_1\, y_1 + x_2 y_2) / (| \mathbf{r_1} | \, | \mathbf{r_2} |). \quad (33)$$

For the two position vectors we have been considering:

$$\cos(\lambda) = (8.7\, \mathbf{i} + 23.8\, \mathbf{j}) \bullet (15.5\, \mathbf{i} - 8.9\, \mathbf{j}) / (| 8.7\, \mathbf{i} + 23.8\, \mathbf{j} | \, | 15.5\, \mathbf{i} - 8.9\, \mathbf{j} |)$$

or, $$\cos\lambda = \frac{(8.7)(15.5) + (23.8)(-8.9)}{\sqrt{8.7^2 + 23.8^2}\sqrt{15.5^2 + (-8.9)^2}} = -0.171$$

and so, $\lambda = 100°$. (34)

The answer, 100°, is obviously the angle between a vector going N20E (Az 20°) and another going S60E (Az 120°). For position vectors in three dimensions, however, the answer is not so obvious.

A three-dimensional problem

Vectors earn their keep in three-dimensional problems – for example, in working out the relative position of locations on multi-level maps. As an illustration, consider the following cave. You enter the cave along an inclined shaft that runs in a direction N43W for a distance of 48 m with a plunge (the slope down from the horizontal) of 38°. At a distance of 28 m down the shaft, a tunnel goes off in a direction of N48E and up an incline of 3°. Call this Level *A*, and let Point *A1* be the start of the tunnel at the shaft. At the bottom of the shaft (48 m down from the entrance), a second tunnel goes off from the shaft. This second tunnel heads in a direction of N21W and up an incline of 6°. Call this Level *B*, and let Point *B1* be the start of the tunnel at the shaft. You explore both tunnels.

Starting at *A1*, the tunnel at Level *A* continues at a constant bearing and incline for 49 m, where it turns (Point *A2*). The new direction is S70W, and the tunnel descends at a plunge of 12° for 36 m to Point *A3*, where the tunnel ends.

Starting at Point *B1*, the tunnel at Level *B* continues at a constant bearing and incline for a distance of 55 m, and then it turns (Point *B2*) and goes for 47 m in a direction of S40E and up an incline of 4°. At this point, Point *B3*, the tunnel ends.

You can probably guess the questions. Where are *A3* and *B3* relative to the entrance of the cave and relative to each other? How far away are they from each other?

You start by defining an origin of an *xyz*-coordinate system. An obvious choice is the entrance to the cave. Let *x* be due east, *y* be due north, and *z* be straight *up*. The procedure then is to convert all of the given information into point-to-point vectors; then add appropriate segments to get the position vectors to *A3* and *B3*; then subtract the two position vectors to get the vector between *A3* and *B3*.

Let **v** be a general vector for which you know its length |**v**|, its azimuth (AZ) in the horizontal plane, and its plunge (PL) downward from the horizontal plane. Then, with a little trig, you can show:

(1) The projection of **v** onto the z-axis is

$$v_z = |\mathbf{v}| \cos(90° + PL)\ ; \tag{35}$$

(2) The projection of **v** onto the *xy*-plane (horizontal) is

$$v_H = |\mathbf{v}| \sin(90° + PL); \tag{36}$$

(3) The projection of **v** onto the positive *y*-axis is

$$v_y = v_H \cos(AZ) = |\mathbf{v}| \cos(AZ) \sin(90° + PL)\ ; \text{ and} \tag{37}$$

(4) The projection of **v** onto the positive *x*-axis is:

$$v_x = v_H \cos(90° - AZ) = |\mathbf{v}| \cos(90° - AZ) \sin(90° + PL). \tag{38}$$

We start with the inclined shaft. Let $\mathbf{v}_{\mathbf{O\rightarrow A1}}$ be the vector from the entrance (Origin) to A1. Then (carrying more digits than we want in the end), the three components are:

$$(\mathbf{v}_{\mathbf{O\rightarrow A1}})_x = 28*\cos(90°- 317°)* \sin(90°+38°) = -15.0$$
$$(\mathbf{v}_{\mathbf{O\rightarrow A1}})_y = 28*\cos(317°)* \sin(90°+38°) = 16.1$$
$$(\mathbf{v}_{\mathbf{O\rightarrow A1}})_z = 28*\cos(90°+38°) = -17.2, \qquad (39)$$

where all the lengths are in meters. Combining the components, the vector is:

$$\mathbf{v}_{\mathbf{O\rightarrow A1}} = -15.1\ \mathbf{i} + 16.1\ \mathbf{j} - 17.2\ \mathbf{k}, \qquad (40)$$

where $\mathbf{k}$ is the unit vector along the positive z-axis. Similarly, the vector from the entrance to B1 is:

$$\mathbf{v}_{\mathbf{O\rightarrow B1}} = -25.8\ \mathbf{i} + 27.7\ \mathbf{j} - 29.6\ \mathbf{k}\ . \qquad (41)$$

You can check these results by checking that the components produce the known magnitude. For example, from the general relation,

$$|\,\mathbf{v}\,| = \sqrt{v_x^2 + v_y^2 + v_z^2}\ , \qquad (42)$$

you have

$$|\,\mathbf{v}_{\mathbf{O\rightarrow B1}}\,| = \sqrt{(-258)^2 + 27.7^2 + (-29.6)^2}\ , \qquad (43)$$

which checks.

By the same process, you can determine the vectors from *A1* to *A2* and from *A2* to *A3*:

$$\mathbf{v}_{\mathbf{A1\rightarrow A2}} = 36.4\ \mathbf{i} + 32.7\ \mathbf{j} + 2.6\ \mathbf{k}\ , \text{ and}$$
$$\mathbf{v}_{\mathbf{A2\rightarrow A3}} = -33.1\ \mathbf{i} - 12.0\ \mathbf{j} - 7.5\ \mathbf{k}\quad . \qquad (44)$$

Similarly:

$$\mathbf{v}_{\mathbf{B1\rightarrow B2}} = -19.6\ \mathbf{i} + 51.1\ \mathbf{j} + 5.8\ \mathbf{k}\ , \text{ and}$$
$$\mathbf{v}_{\mathbf{B2\rightarrow B3}} = 30.1\ \mathbf{i} - 36.0\ \mathbf{j} + 3.3\ \mathbf{k}. \qquad (45)$$

Now add up the appropriate vectors. For the position vector $\mathbf{r}_{\mathbf{A3}}$ from the origin at the entrance of the cave to *A3*:

$$\mathbf{r}_{\mathbf{A3}} = \mathbf{v}_{\mathbf{O\rightarrow A1}} + \mathbf{v}_{\mathbf{A1\rightarrow A2}} + \mathbf{v}_{\mathbf{A2\rightarrow A3}}$$
$$= (-15.0 + 36.4 - 33.1)\ \mathbf{i} + (16.1 + 32.7 - 12.0)\ \mathbf{j} + (-17.2 + 2.6 - 7.5)\ \mathbf{k}\ ,$$
$$= -11.7\ \mathbf{i} + 36.8\ \mathbf{j} - 22.1\ \mathbf{k}, \qquad (46)$$

which means that *A3* is 11.7 m west of the cave entrance, 36.8 m north of it, and 22.1 m below it. By similar arithmetic, the position vector $\mathbf{r_{B3}}$ is:

$$\mathbf{r_{B3}} = -15.3\ \mathbf{i} + 42.8\ \mathbf{j} - 20.5\ \mathbf{k}. \qquad (47)$$

The vector from *A3* to *B3* is found by subtracting $\mathbf{r_{A3}}$ from $\mathbf{r_{B3}}$. Thus:

$$\mathbf{v_{A3\rightarrow B3}} = \mathbf{r_{B3}} - \mathbf{r_{A3}} = -3.6\ \mathbf{i} + 6.0\ \mathbf{j} + 1.6\ \mathbf{k}, \qquad (48)$$

which means that *B3* is 4 m west, 6 m north, and 2 m up(!) from *A3* (so much for the two "levels"). The straight-line distance between the two points is:

$$|\ \mathbf{v_{A3\rightarrow B3}}\ | = \sqrt{(-3.6)^2 + 6.0^2 + 1.6^2} = 7.2 \qquad (49)$$

or 7 m. The direction θ (in the horizontal plane, clockwise from the *x*-axis) is found from the ratio of the *y*- and *x*-components:

$$\theta = \arctan\left(\frac{6.0}{-3.6}\right) = \arctan(-1.667)\,. \qquad (50)$$

The two possible answers are $\theta = -59°$ and $\theta = -239°$ corresponding to AZ = 149° (S31E) and AZ = 329° (N31W), respectively. The latter is the appropriate choice. Finally, the plunge is found from the dot product of $\mathbf{v_{A3\rightarrow B3}}$ and the projection of $\mathbf{v_{A3\rightarrow B3}}$ onto the *xy*-plane (you get this projection by simply omitting the *z*-component of the sloping vector):

$$\cos(\lambda) = (-3.6\mathbf{i} + 6.0\mathbf{j} + 1.6\mathbf{k}) \bullet (-3.6\mathbf{i} + 6.0\mathbf{j}) \,/\, (\,|\,-3.6\mathbf{i} + 6.0\mathbf{j} + 1.6\mathbf{k}|\ |\,-3.6\mathbf{i} + 6.0\mathbf{j}\,|\,), \qquad (51)$$

from which the plunge works out to be 13°.

Vectors from the center of the Earth

As our last example, we will consider a question that you might not think to solve with vectors: How far is it between, say, Seattle and Miami? Here, "how far" means the shortest distance *on*, as opposed to *through*, the sphere of the Earth. In other words, what is the length of the great-circle arc between the two points? All you need to know is the radius of the Earth (6,370 km, to three significant digits), the latitude and longitude of Seattle (47.35N, 122.20W) and Miami (25.45N, 80.15W), the meaning of latitude and longitude in terms of coordinates, and a little about vectors. Specifically, what you need to know about vectors is that the dot product gives you the angle between two vectors.

Conceptually, the key to the problem is that the latitude and longitude of a point on the globe allow you to define a position vector from the center of the Earth to the point. You merely need to convert the latitude and longitude information to Cartesian coordinates. To do this, define a (right-handed) coordinate system with the origin at the center of the Earth (i.e., a geocentric coordinate system that rotates with the Earth). Let the *z*-axis run up through the North Pole; let the *x*-axis pass through the equator at the Prime (or Greenwich) Meridian (i.e., where latitude = 0 and longitude = 0); and let the *y*-axis pass through the equator at longitude 90°

E (so that the y-axis is 90° counterclockwise from the x-axis as viewed looking down the z-axis). By the convention that angles in the xy-plane are positive counterclockwise from the x-axis: east longitude is positive, and west longitude is negative. Similarly, latitude is positive in the Northern Hemisphere and negative in the Southern Hemisphere. Then, the angular information together with the radius of the Earth gives three coordinates that specify the location of points on the surface of the Earth. For our two points of interest:

Seattle: r = 6,370 km, $\Theta_{lat} = 47.35^{o}$, $\Theta_{long} = 122.20^{o}$W
Miami: r = 6,370 km, $\Theta_{lat} = 25.45^{o}$, $\Theta_{long} = 80.15^{o}$W

How do these latitudes and longitudes translate into the geocentric Cartesian coordinates?

The z-coordinate is the projection of the position vector (from the center of the Earth) onto the z-axis and is given by

$$z = r \sin (\Theta_{lat}). \tag{52}$$

The projection of the position vector onto the equatorial plane (the xy-plane) of the Earth is

$$r_{\mathrm{H}} = r \cos (\Theta_{lat}). \tag{53}$$

Then, the x-coordinate is the projection of this equatorial projection (r_{H}) onto the x-axis:

$$x = r \cos (\Theta_{lat}) \cos (\Theta_{lat}). \tag{54}$$

Similarly, the y-coordinate is the projection of r_{H} onto the y-axis:

$$y = r \cos (\Theta_{lat}) \sin (\Theta_{lat}). \tag{55}$$

With these equations, the geocentric Cartesian coordinates of the two points are:

Seattle: $x = -2{,}300$, $y = -3{,}652$, $z = 4{,}685$
Miami: $x = 984$, $y = -5{,}667$, $z = 2{,}737$,

all in kilometers. So, you can immediately write down the two position vectors:

$$\mathbf{r}_{\mathbf{Seattle}} = -2300\ \mathbf{i} - 3652\ \mathbf{j} + 4685\ \mathbf{k}$$
$$\mathbf{r}_{\mathbf{Miami}} = 984\ \mathbf{i} -5667\ \mathbf{j} +2737\ \mathbf{k}. \tag{56}$$

The question now is: What is the angle between the two vectors of equations 56? The equation to use is equation 33, which for this problem becomes:

$$\cos (\lambda) = (\mathbf{r}_{\mathbf{Seattle}} \bullet \mathbf{r}_{\mathbf{Miami}}) / r_{\mathrm{Earth}}^{2}. \tag{57}$$

Then, multiplying out as in equations 34, we get:

$$\cos (\lambda) = [(-2300)(984) + (-3652)(-5667) + (4685)(2737)] / 6370^{2} = 0.770$$

$$\lambda = 39.6° \, . \tag{58}$$

To complete the solution, we need only to calculate the length on a spherical Earth that corresponds to a geocentric angle of 39.6°. In other words: How long is 39.6/360 (or 11.0%) of the Earth's circumference? The Earth's circumference is 40,000 to three significant figures. So, the great-circle distance from Seattle to Miami is 11% of 40,000 km, or 4,400 km to three significant figures.

Not that there would be any reason other than simple curiosity, but we can also calculate the straight-line distance from Seattle to Miami (i.e., the length of the shortest tunnel between them). First, subtract the two position vectors to obtain the vector that runs from Seattle straight to Miami:

$$\begin{aligned} \mathbf{r_{Mia}} - \mathbf{r_{Sea}} &= (984+2300)\,\mathbf{i} + (-5667+3652)\,\mathbf{j} + (2737-4685)\,\mathbf{k} \\ &= 3284\,\mathbf{i} - 2015\,\mathbf{j} - 1948\,\mathbf{k} \, . \end{aligned} \tag{59}$$

Then apply equation 42 (the three-dimensional Pythagorean relation) to find the magnitude of the vector of equation 59:

$$|\, \mathbf{r_{Mia}} - \mathbf{r_{Sea}} \,| = \sqrt{3284^2 + (-2015)^2 + (-1948)^2} = 4317 \, , \tag{60}$$

or 4,320 km to three significant figures.

In passing, a historical note is appropriate. It is no coincidence that the circumference of the spherical Earth came out to be the nice round number, 40,000 km. The 40,000-km circumference makes the length of a quadrant of a meridian (i.e., the great-circle distance from the pole to the equator) equal to 10,000 km. It is that length, because the meter was defined to make it so. The commission that defined the metric system was interested in units that related to what were perceived as natural, fundamental quantities. For length, they selected the circumference of the Earth as a fundamental quantity and defined the meter to be $1/10^6$ of a quadrant of that length. This was shortly after the French Revolution, and the panel included such historical figures as Legendre and Laplace, of mathematical fame, and Lavoisier, the so-called father of chemistry. This definition set into motion efforts to determine the length of a meridian (or the length of a degree of latitude), which in turn led to discoveries about the shape of the Earth and the deflection of plum bobs near large masses (i.e., isostasy). As a result of this original metric-scale definition of the meter, it is easy to remember the size of the Earth. The circumference is $4 \times 10{,}000$ km; the radius is this value divided by 2π (or 6.37×10^3 km to three significant figures).

Concluding remarks

You learn vectors in your freshman physics course. But don't leave them there. They can help you with the most basic of geography and geology problems. For example, how else would you get the shortest distance between two points on the surface of the Earth? Don't say, "use a map", because that won't work (unless you mean a globe). Most people I ask don't know; some tell me there are places on the Web to go, and some mention their GPS system. But if **you** need an answer to this or any other problem involving distance and direction, use the trusty vectors.