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Computational Geology 3

Progressing Geometrically

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Introduction

The key point of "Speaking Logarithmically," the Computational Geology column of the last issue, was that logarithms are exponents. As such, they allow multiplication and division to be replaced by the easier addition and subtraction. That replacement was the goal of John Napier when he invented logarithms; however, he was not thinking of exponents.

Napier (1550-1617) was a baron who lived in Merchiston Castle outside of Edinburgh, Scotland. By his account, his most important undertaking was a book (1593) in which he used the methods of Euclidean deduction to argue that the Pope was the Antichrist. In the process, he predicted that the world would end in 1786. After that, he worked 20 years to develop logarithms, which he published in two books. The first to appear, but second to be written, was *Mirifici Logarithmorum Canonis Descriptio* (Description of the Wonderful Canon of Logarithms, 1614). The second, which appeared posthumously, was *Mirifici Logarithmorum Canonis Constructio* (Construction of the Wonderful Canon of Logarithms, 1619).

Characterizing logarithms as "wonderful" may seem strange to present-day students, but you must realize that logarithms completely revolutionized calculations. Moreover, you may begin to appreciate Napier's genius when you realize that this was at a time when the use of zero was not firmly established, and when decimal fractions had not been invented.

Most interesting, I believe, is the fact that Napier had no concept of exponents. Indeed exponents, in the form of x^3 for example, were not invented until Descartes' *La géométrie* (1637). It was Euler in 1748 who first published the concept of logarithms as exponents.

What did Napier have in mind for logarithms if they were not exponents? More to the point of this chapter: So what? Is it just a mathematical-historical curiosity, or can Napier's concept of logarithms help you be a more mathematically skilled geologist? I believe the answer to this last question is, "Yes!"

The key to Napier's concept is that logarithms bridge between geometric progressions and arithmetic progressions. Understanding this connection leads to a powerful way of looking at semilog and log-log graphs, including those involving geological variables.

Geometric Progressions and Logarithms.

The word for a succession of numbers for which there is a rule relating one to the next is *sequence*. If the terms are added together, the succession is a *series*. Commas separate the terms of a sequence. Pluses and/or minuses connect the terms of a series. Geometric and arithmetic progressions, the subject of this column, are sequences.

To go from one term to the next in a *geometric progression* (*GP*), you simply multiply by a number, r, which is the *common ratio* (*c.r.*) between terms in the progression. So, the general form of a geometric progression is:

$$a, ar, ar^2, ar^3, ar^4, \ldots.$$

If $n = 0$ for the first term, then each term is given by ar^n. An example (with $a = 1$ and $r = 2$, to $n = 4$) is: 1, 2, 4, 8, 16. Another example (with $a = 5$ and $r = 3$, to $n = 4$) is: 5, 15, 45, 135, 405.

To go from one term to the next in an *arithmetic progression* (*AP*), you add a number, d, which is the *common difference* (*c.d.*) between terms in the progression. So, the general form of an arithmetic progression is:

$$b, b + d, b + 2d, b + 3d, b + 4d, \ldots.$$

If $n = 0$ for the first term, then each term is given by $b + n*d$. An example (with $b = 0$ and $d = 1$, to $n = 4$) is 0, 1, 2, 3, 4. Another example (with $b = 5$ and $d = 3$, to $n = 4$) is: 5, 8, 11, 14, 17.

The theme of this column is: What happens when you pair off progressions? For example, consider the following pair:

GP	1	2	4	8	16	32	64
AP	0	1	2	3	4	5	6

The *GP* might represent the number of individuals of a particular species (where the numbers are in thousands, say), with the *AP* representing years from the initiation of population counting. The *GP* shows *geometric growth* of this population. The *c.r.* of the *GP* is 2, and it corresponds to a *c.d.* of 1 in the *AP*. That means, for this example, the population doubles every year.

It is a small step to extend this progression to the left, thereby picking up negative numbers in the *AP* and fractions in the *GP*. Thus:

GP	.0625	.125	.25	0.5	1	2	4	8
AP	-4	-3	-2	-1	0	1	2	3

Still, the *c.d.* of the *AP* is 1, and the *c.r.* of the *GP* is 2.

The point to logarithms is this: If the *GP* has $a = 1$, and the *AP* has $b = 0$, and the two progressions are paired off so that $GP = 1$ corresponds to $AP = 0$ – as is the case in our example – then the terms of the *AP* are produced from the terms of the *GP* by:

$$AP = d \log_r (GP)$$

In our example, the *AP* is produced by $\log_2(GP)$ because $d = 1$.

Now suppose the *GP* is turned around using as a pivot the column where $AP = 0$ and $GP = 1$:

GP	8	4	2	1	.5	.25	.125	.0625
AP	-3	-2	-1	0	1	2	3	4

The *c.d.* of the *AP* is still 1, but the *c.r.* of the *GP* is now 1/2. In other words, you reverse the order when you change the *c.r.* from r to $1/r$. In this case, $AP = \log_{1/2}(GP)$. You can see by comparing the two tables that this is the same as $AP = -\log_2(GP)$, which you can show directly by using the rule for changing bases (Rule 4 in the previous "Computational Geology").

Phi Sizes

The last pairing of an *AP* and *GP* is part of the phi (ϕ) size scale that you had (or will have) in your sediments course. Recall that the grain size of sedimentary particles ranges from boulders on down to clay-size particles. According to a nomenclature proposed by Wentworth (1922), boulders are particles with a diameter larger than 256 mm, and clay sizes are smaller than 1/256 mm across; sand sizes range from 2 mm to 1/16 mm. This nomenclature became known as the Wentworth Scale, and, perspicaciously, all the boundaries between sizes were powers of 2. This fact enabled Krumbein (1936) to get rid of all those fractions of millimeters by defining a phi size as:

$$\phi = -\log_2(D_{mm})$$

where D_{mm} is the grain diameter in mm.

(Without intending to go too far afield with ϕ sizes themselves, I should point out that D_{mm} needs to be thought of in the right way. The argument of the log function needs to be dimensionless, and so D_{mm} must be considered not a length, but "just a number" – the number of times longer than a millimeter. If the argument of the function is considered a length, then the user is free to choose units. An entirely messy situation would arise, obviously, if different users used different units. McManus [1963] faced the distinction explicitly by asking that the argument of the log function be considered the ratio of the grain diameter, D, to a standard diameter, D_0, which he chose as 1 mm. By either approach, the argument of the log transformation producing ϕ is somewhat akin to specific gravity, a dimensionless variant of density. McManus's formulation did not catch on. As another sidelight: the very first paper published in *Geology*, Shea [1963] proposed a base-ten log transformation of a McManus ratio with several advantages over the base-two approach. That idea did not catch on either.)

A boulder-to-clay listing of D_{mm} vs. ϕ sizes, for whole ϕ's, is shown in Table 1. Column 1 gives the diameters, and column 2 gives the ϕ sizes. The columns are not labeled as "D_{mm}" and "ϕ" because we are going to do something else with them. For our purposes, Table 1 is simply a listing of a *GP* with $a = 1$ and $r = 1/2$ in column 1 lined up with an *AP* with $b = 0$ and $d = 1$ in column 2. Because the *c.r.* of the *GP* is a fraction, the numbers in it decrease as the numbers in the *AP* increase. (Krumbein defined ϕ with the minus sign so that large sizes would be on the left side and small sizes would be on the right side of the graph when you plot frequency vs. grain size using ϕ for the grain size.)

Now here comes the point relating to Napier's breakthrough: Multiply 64×0.0078125. Don't write it out; look at Table 1. Find the first number (64) in the *GP*, and note the corresponding number (–6) in the *AP*; similarly, find the second number (0.0078125) in the *GP*, and note its corresponding number (+7) in the *AP*. Add the two numbers of the *AP* together (–6 + 7), and move to the result (+1) in the *AP*. Then find the corresponding number (0.5) in the *GP*. That's your answer! Division works similarly, with subtraction in the *AP* replacing division in the *GP*.

Column 1	Column 2
256	−8
128	−7
64	−6
32	−5
16	−4
8	−3
4	−2
2	−1
1	0
0.5	1
0.25	2
0.125	3
0.0625	4
0.03125	5
0.015625	6
0.0078125	7
0.00390625	8

Table 1. Phi Sizes, A Tool for Multiplication and Division

In short, addition and subtraction of numbers in the *AP* corresponds to multiplication and division, respectively, of numbers in the *GP*. Our table is thus a tool to multiply and divide numbers. That is because the numbers in the *AP* are logarithms of the numbers in the *GP*. *When Napier paired a GP with an AP for the purpose of transforming multiplication and division into addition and subtraction, he invented logarithms.*

What Napier Actually Did

Napier did not use a *GP* with a *c.r.* of 1/2, as we have done in Table 1. The reason why such a table would not be useful can be illustrated simply: Using Table 1, what is 0.567 × 11.5? The gaps between the numbers in the *GP* of Table 1 are clearly too large. In order to have a table with steps small enough that the table could be of use, Napier needed to have a *c.r.* very close to 1. The number he chose was 0.9999999. Unfortunately for the smooth telling of the story some four hundred years later, Napier did not start the *GP* of his table with 1. Rather, he started it with 10,000,000. Thus, the first four rows of what we will call "the Napier table" are:

Col. 1	Col. 2
10000000	0
9999999	1
9999998	2
9999997	3

To Napier, column 2 was the number of times 10,000,000 needs to be multiplied by .9999999 to produce the number in column 1. Thus column 1 of the Napier table is a *GP* with $r = 1-10^{-7}$, in modern notation; column 2 is a count, or an *AP* with $d = 1$.

Had Napier started his *GP* with 1 instead of 10,000,000, the first four terms would have been:

Col. 1	Col. 2
1.0000000	0
.9999999	1
.9999998	2
.9999997	3

From this listing and the previous illustration of ϕ sizes, you can see that the numbers of column 2 of *this table* are simply the logarithms to the base 0.9999999 of the numbers of column 1. In other words, the numbers of column 2 of *the Napier table* are the logarithms to the base 0.9999999 of the numbers (N) in column 1 divided by 10^7. Or, symbolically, in the Napier table:

$$\text{Napier table, column } 2 = \log_{0.9999999}(N/10^7).$$

Napier called these numbers logarithms, taking the word from Greek roots meaning ratio and number. To Napier, the numbers in the second column were "*ratio numbers*", because the differences between them depended only on the ratio of the corresponding numbers of the first column.

Comparison of these tables shows that there is a slight disparity between what we call logarithms today and the original logarithms. Also, because of the 10^7, the rules for multiplication and division do not work quite the way they should by our present standards. These quibbles involve only the location of the decimal point, a concept that was not established at the time.

To avoid leaving the wrong impression, I need to add that what I have called "the Napier table" was only the first of three tables in Napier's canon. This first table, which was a *GP* with $r = (1-10^{-7})$, ran to $n = 100$. The second table, which ran to $n = 50$, was a *GP* with $r = (1-10^{-5})$. The main table had 21 rows and 69 columns with the rows having $r = (1-1/100)$ and columns having $r = (1-1/2000)$. These tables were intended to be used in combination, with the first two providing a means of interpolating within the third.

Natural logarithms

As you probably know, Napier is credited with inventing natural logarithms. Natural logarithms have the base *e*. But Napier had no concept of bases, and it was Euler who discovered *e*. So, how could Napier's logarithms be our natural logarithms?

The short answer is that they are not. But they are very close. To see this, we can use the relationship that we used earlier to turn the ϕ sizes around:

$$\phi = -\log_2(D_{mm}) = \log_{1/2}(D_{mm})$$

Thus Column 2 of the Napier table can also be expressed:

$$\text{Napier table, column } 2 = -\log_{(1/0.9999999)}(N/10^7)$$

Now look at that base, 1/0.9999999. It is the same as 1.0000001 (or $1+10^{-7}$). You can convince yourself of this by using the binomial series (discovered by Newton in 1665, some fifty years after Napier's logarithms). (Or you can simply use your calculator.) The result, then, is that Napier's logarithms can be written:

$$\text{Napier table, column 2} = -\log_{1.0000001}(N/10^7)$$

Now we need to look at Euler's *e*. Euler found that *e*, which he defined from an infinite series, could be expressed also as the limit as *n* becomes infinitely large of $(1+1/n)^n$. He calculated *e* to 23 places:

$$e = 2.71828182845904523536028.$$

How does this number for a base relate to Napier's base of $(1+1/10^7)$? To start, we can note that, although 10^7 is not infinite, it is certainly very large, and so Napier's base raised to the 10^7 power should be quite close to *e*. Try it on your calculator; mine gives

$$(1+1/10^7)^{10{,}000{,}000} = 2.7182817$$

So, to seven figures, we can say that *e* can be expressed as $(1.0000001)^{10{,}000{,}000}$.

To complete the exercise, we need only to convert the logs with a base of 1.0000001 to logs with a base of $1.0000001^{10{,}000{,}000}$. Remember (from the Computational Geology 2):

$$\log_{new\ base}(U) = \log_{old\ base}(U) / \log_{old\ base}(new\ base)$$

So, let *old base* = 1.0000001 and *new base* = $1.0000001^{10{,}000{,}000}$. Then:

$$\log_{1.0000001^{10{,}000{,}000}}(U) = \frac{\log_{1.0000001}(U)}{\log_{1.0000001}(1.0000001^{10{,}000{,}000})}.$$

Now substitute for *e* as the base on the left side and recall Euler's definition of logarithms as exponents for the denominator on the right side; you get

$$\log_e(U) = \frac{\log_{1.0000001}(U)}{10^7}.$$

Finally, recognize the numerator on the right side as the negative of column 2 of the Napier table. Thus, in modern terminology, the columns of the Napier table could be labeled:

N	$-10^7 \log_e(N/10^7)$
10000000	0
9999999	1
9999998	2
9999997	3

This is close enough to say that Napier discovered natural logarithms. Euler put them on the modern footing.

Other Players

Logarithms were independently discovered by Jobst Bürgi (1552-1632), a Swiss watchmaker, who published his work in 1620. His *AP* was $10*n$ and, starting with 0, ran to n = 23,027. His *GP* was $10^8(1+10^{-4})^n$ and started with 10^8. Therefore, he paired a *GP* with r = 1.0001 to an *AP* with d = 10. Because the *c.r.* was greater than 1, the numbers of both progressions increased with increasing n (unlike Napier's tables). Because $1.0001^{23,027}$ is 9.9999979 (close to 10), the last entry in the *AP*, – i.e., 230,270 – corresponded to 1,000,000,000 in the *GP*. Comparing those numbers to 2.302585 = ln(10), you can see that Bürgi, like Napier, came to within placement of the decimal point of getting what we know as natural logarithms. They both had the same concept: counting off a *GP-AP* pair. Appropriately, the title of Bürgi's book was *Arithmetische und geometrische Progress-Tabulen*.

Henry Briggs (1561-1630) was a professor of geometry at Oxford. In 1615 he traveled to Merchiston Castle, where he and Napier concluded that the table of logarithms would be improved if they paired 10 in the *GP* with 1 in the *AP*, in addition to Napier's (and Bürgi's) pairing of 1 in the *GP* with 0 in the *AP*. Because the pairing would thus be anchored at two points, the *AP* and *GP* needed to be created a different way. Briggs did it by determining successive square roots and then applying the rule of multiplication (that logarithms add). He published new tables of logarithms in 1617 and 1624. The two anchors predetermined the base to be 10. The resultant "common logarithms" are sometimes called "Briggsian logarithms".

Finally, William Oughtred (1574-1660), an English vicar and one of the most influential teachers of mathematics of the time, needs to be included in any account of logarithms. Oughtred is credited with developing the slide rule, a computational instrument dear to people of my pre-calculator generation. There is no better example of a *GP-AP* pairing. Consider:

GP	1	3.16	10	31.6	100
GP	1	3.16	10	31.6	100
AP	0	2.5	5.0	7.5	10.0

Think of the *AP* as the number of inches from the left-hand side of the rule. The two *GP*'s are the numbers indicated on the rule at the lengths of the *AP*. One *GP* can slide against the other. Then: slide the lower *GP* to where its "1" lines up against the "3.16" of the upper *GP*; look at the "10" of the lower GP in its new position; see that “10" resting against "31.6" of the upper GP. That “31.6" is the result of multiplying 3.16 by 10. Effectively, this sliding operation adds 2.5 inches to 5.0 inches to simulate the multiplication of 3.16 by 10. It does mechanically, and with common logarithms, what we did with the ϕ-scale or base-1/2 logarithms of Table 1.

How are natural logarithms "natural"?

The story of Napier and Bürgi should partly answer the question of what is natural about natural logarithms. Natural logarithms can be generated "naturally" by setting one anchor and counting off the two progressions. This is what Napier and Bürgi did, and they came within placement of the decimal point of getting them. On the other hand, Briggs's logarithms with the two anchors involved an arbitrary decision – i.e., log(10) = 1. Presumably this decision was influenced by the number of fingers on our hands. The base did not just arise "naturally".
There is, however, a bigger reason that natural logarithms are "natural." It has to do with the rate

that the *GP* changes relative to the rate that the paired-off *AP* changes. The point can be seen in Figure 1, which shows a spreadsheet printout with rows and columns labeled as spreadsheets are. Column B is simply a count (n), and column C is the *GP*, with $r = 1 + 10^{-7}$, produced from r^n. Note that the spreadsheet gives only a few rows from $n = 0$ to $n = 10{,}000{,}002$.

	B	C	D	E	F	G	H
	n	*GP*	difference	*AP1*	difference	*AP2*	difference
5	0	1		0		0	
6	1	1.0000001	1E-07	1E-07	1E-07	4.34294E-08	4.34E-08
7	2	1.0000002	1.0000001E-07	2E-07	1E-07	8.68589E-08	4.34E-08
8							
9	1,000,001	1.105171023		0.100000095		0.043429489	
10	1,000,002	1.105171134	1.1051710E-07	0.100000195	1E-07	0.043429533	4.34E-08
11							
12	5,000,001	1.648721395		0.500000075		0.217147274	
13	5,000,002	1.64872156	1.6487214E-07	0.500000175	1E-07	0.217147317	4.34E-08
14							
15	10,000,001	2.718281966		1.000000051		0.434294504	
16	10,000,002	2.718282238	2.7182820E-07	1.000000151	1E-07	0.434294547	4.34E-08

Figure 1. Spreadsheet illustrating property of logarithms. Column B lists selected numbers between 0 and 10,000,002. Columns C through H all contain equations. Representative cell equations are: for C10: = (1.0000001)^B10; for D10: =C10–C9; for E10: =ln(C10); for F10: =E10–E9; for G10: =log(C10); for H10: =G10–G0.

In passing, note that Column B and C are similar in spirit to the *AP* and *GP* of Napier and Bürgi. Like Napier's pairing, the small fractional departure from 1 in the *c.r.* is 10^{-7}. Like Bürgi's pairing, the small fraction is added to rather than subtracted from 1, so both progressions increase together.

The main feature to notice is Column D, the difference between successive numbers in the *GP* of Column C. Note that *except for the order of magnitude* (10^{-7} in Column D), and to seven places, the *differences between successive numbers are the same as the numbers themselves* (e.g., using cell numbers, cell D10 is $C9 * 10^{-7}$; or, $D10/C9 = 10^{-7}$).

The rest of Figure 1 pertains to two *AP*'s generated from the *GP* of Column C. *AP1* (Column E) is the natural logarithm. *AP2* (Column G) is the common logarithm. As you can see, the differences between successive numbers of the *AP*'s run constant down both columns (that's why they are *AP*'s). Of immense significance is the fact that the *difference in successive numbers of AP1 is* 10^{-7} *– the ratio of (1) the difference between successive numbers of the GP to (2) the number in the GP*. For example, D10/C9 is the same as F10; a similar relationship holds for those three columns down the entire figure. In contrast, column H, the difference between the *AP* generated by the base-10 logarithm, is "off" the 10^{-7} (e.g., D10/C9) by a factor, 0.434.

There is *only one base* that produces the simple relationship between the *GP* column and the two difference columns, and that's the base equal to the extraordinary number *e*. If we had somehow evolved to have eight digits on each hand and had come up with a hexadecimal number system (Computational Geology 2) and a hexadecimal base for logarithms, there would have been a hexadecimally-digited Euler who would have come up with a hexadecimally expressed *e* for the natural logarithm, the function that connects Columns C, D, and F of Figure

1.

Of course, if you know calculus, you know why all this is so. Let the numbers of column C be *x*, and let the numbers of Column E be *y*. Then Columns D and F are *dx* and *dy*, respectively. The relationship between Columns C, D, E and F is $dx/x = dy$ with $y = 0$ at $x = 1$. There is only one function that can produce that relationship, and that is ln(*x*) (which is *AP1*).

By the way, 1/ln(10) = 0.4342945, the factor by which Column H is off the 10^{-7}. If you know calculus, you know why that is too.

Graphs

The relationship between progressions and graphs can be stated very simply. If the *y* variable progresses *arithmetically* while the *x* variable progresses *arithmetically*, then *y* vs. *x* will plot as a straight line on *arithmetic graph paper* (the usual kind). If the *y* variable progresses *geometrically* while the *x* variable progresses *arithmetically*, then log(*y*) to any base vs. *x* will plot as a straight line on arithmetic graph paper; this is the same as saying that *y* vs. *x* will plot as a straight line on *semilog graph paper* if *y* is plotted on the log scale. Finally, if the *y* variable progresses *geometrically* while the *x* variable progresses *geometrically*, then log(*y*) vs. log(*x*) will plot as a straight line on arithmetic graph paper; this is the same as saying that *y* vs. *x* will plot as a straight line on *log-log paper*. We will consider an example of each.

Pairing two arithmetic progressions. As an example of an *AP* paired against another *AP*, consider:

n	0	1	2	3	4
AP1	0	25	50	75	100
AP2	32	77	122	167	212

You may recognize this as a succession of temperatures in °C (*AP1*) paired with equivalent temperatures in °F (*AP2*). If you plot these values of *F* (temperature in °F) against the corresponding values of *C* (temperature in °C), you will get a straight line for *F* vs. *C*.

Bringing in the counter *n*, you can see that $F = 25n$ and $C = 32 + 45n$; that is, 25 is the *c.d.* for *AP1*, and 45 is the *c.d.* for *AP2*. Now, solve for *n* in the equation for *F* (i.e., $n = F/25$), and substitute the result into the equation for *C*. This gives:

$$F = 32 + \frac{9}{5}C\,,$$

which is the equation to convert °C to °F. It is a *linear function*, the general form of which is $y = \alpha + \beta x$. Note that the slope of the line (9/5) is the *AP2*-to-*AP1* ratio of the *c.d.*'s (45/25).

Because the slope is 9/5, you can easily work out how much *F* changes if *C* changes by some amount other than 5. Not only does *F* change by 9° when *C* changes by 5°. *F* changes also, for example, by 12.6° when *C* changes by 7° (because 12.6/7 = 9/5). That means the pairing

n	0	1	2	3	4
Ap3	0	7	14	21	28
AP4	32	44.6	57.2	69.8	82.4

is another pairing representing the relationship between *F* and *C*.

Pairing a geometric progression and an arithmetic progression. As an example of a *GP* paired against an *AP*, consider:

n	0	1	2	3	4
GP	16	8	4	2	1
AP	0	5,710	11,400	17,100	22,800

Here the *GP* is the radioactivity of a sample of fossil wood in *counts* (beta particles) on a Geiger counter per minute per gram of carbon (assuming 1950, pre-nuclear test values for the starting point). The *AP* is the age of the sample and is given in multiples (*n*) of the half-life ($t_{1/2}$) of ^{14}C (to three figures). If you plot values of log(*GP*) (log counts) vs. values of *AP* (age), you will get a straight line.

Bringing in *n*, you have counts = $16*(1/2)^n$ and age = $5710*n$; that is, 1/2 is the *c.r.* for the *GP*, and 5710 is the *c.d.* of the *AP*. Now solve for *n* in the equation for age (n = age/5710) and substitute the result into the equation for counts. You get:

$$counts = 16\left(\frac{1}{2}\right)^{\frac{age}{5710}},$$

which is an *exponential function*, the general form of which is $y = \alpha\beta^x$. Taking logs of both sides, you get:

$$\log(counts) = \log(16) + \left(\frac{\log(1/2)}{5710}\right) * age$$

which is the equation of a straight line with slope log(1/2)/5710. So, the slope of the line on log paper is the ratio of the log of the *c.r.* of the *GP* to the *c.d.* of the *AP*.

Because the slope is known (it works out to -5.267×10^{-5} yr^{-1}), you can easily determine the length of time for a third-life, say, or a tenth-life, or any other fraction-life. For a tenth-life, we have

$$\frac{\log(1/2)}{5{,}710} = \frac{\log(1/10)}{19{,}000}$$

and so $t_{1/10}$ is 19,000 yrs. This means a plot of ^{14}C radioactivity vs. time on semilog paper (with counts on the log scale) will cross a log cycle for a Δtime increment of 19,000 yrs. It also means that another way of representing ^{14}C decay with a *GP* in counts against an *AP* in years is:

n	0	1	2	3
GP	16	1.6	0.16	0.016
AP	0	19,000	38,000	57,000

Pairing two geometric progressions. Finally, for a *GP* paired against another *GP*, consider

n	0	1	2	3
GP1	0.00400	0.00120	0.00360	0.0108
GP2	0.00148	0.0133	0.120	1.08

Note the *c.r*. of *GP1* is 3, and the *c.r*. of *GP2* is 9. Here, *GP1* is the diameter (*D*) of a small sphere in cm, and *GP2* is the settling velocity (*u*) in cm/sec as calculated from Stoke's Law, assuming ρ_s (density of the sphere) is 2.70 g/cm^3, ρ_f (density of the water) is 1.00 g/cm^3, μ (viscosity of the water) is 0.0100 poise, and *g* is 980 cm/sec^2.

Stokes' Law is one of many examples in geology of a *power function*, the general form of which is $y = \alpha x^{\beta}$. Power functions plot as straight lines on log-log paper. By taking logs of both sides,

$$\log(y) = \log(\alpha) + \beta \log(x)$$

you can see that β, the exponent in the power function, is the slope of the line on the log-log paper. So, a plot of our values of *u* vs. *D* on log-log paper will be a straight line with a slope of 2. This is because of the D^2 in the equation. This relationship between the exponent in the power function and the slope of a line on log-log paper is extremely useful in empirical studies, and particularly in quantitative geomorphology.

The power function will be taken up in a future column (Computational Geology 8). The point here is that the slope, β, of the log-log plot is the ratio of the logs of the *c.r.*'s of the paired-off *GP*'s. In this example:

$$\frac{\log 9}{\log 3} = \frac{\log(3^2)}{\log(3^1)} = \frac{2\log 3}{\log 3} = 2$$

This is the geometric-progression analog of saying that the ratio of *c.d.*'s of paired-off *AP*'s gives the slope of the graph on arithmetic paper.

To end this section on graphs, here is a problem for you: How can you plot settling velocity (from Stoke's Law) vs. grain size in ϕ units to get a straight line, and what would the slope of the line be?

Concluding Remarks

Historically, *GP-AP* pairs were a gateway to logarithms, which provided a breakthrough for calculation. Today, logarithms are not needed, like they once were, for calculation. However, the concept of pairing progressions will forever facilitate interpretation of common graphs.

When you look at a graph, the first thing you should do is look at the axes. Second, you should note the scale on each axis – is it logarithmic or arithmetic? Finally, look at the plot itself. If the plot is a straight line on arithmetic paper, then for every increase of a certain *amount* in one variable, the other variable increases (or decreases) by an *amount* specified by the slope. If the plot is a straight line on semilog paper, then one variable increases (or decreases) by a certain *factor*, given by the slope, for every increase of a certain *amount* in other variable. If the plot is a straight line on log-log paper, then for every increase by a certain *factor* for one variable, the other variable increases (or decreases) by a *factor*, again specified by the slope of the line on the graph.

Sources and Further Reading

The role of pairing *GP*'s with *AP*'s in the development of logarithms is a common denominator of discussions of the history of logarithms. Of particular help to me were the chapters "Napier's Wonderful Logarithms" and "The Age of Euler" in Edwards (1979). Additional historical details were from the ever-useful NCTM (1989) and the classic text by Boyer and Merzbach (1991). In addition I recommend two papers in Swertz et al. (1995): "Revisiting the History of Logarithms" by John Fauvel, and "Napier's Logarithms Adapted for Today's Classroom" by Victor J. Katz. The information to generate the ^{14}C radioactivity table was from Emiliani (1992), which is a great source, in general, for quantitative material about geology.

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