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Computational Geology 19

Classification and the Combination of Sets

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Topics this issue-
Mathematics: Complement, intersection, union; partitions; De Morgan's Laws.
Geology: Classification of rocks; aphanitic vs. phaneritic

Prerequisite: CG-10, "The Algebra of Deduction," Mar 2000; CG-18, "Definition and the Concept of Set," Sept. 2001.

Introduction

The special issue of this journal (Nov. 2000) that was devoted to "Some Great Ideas for Geoscience Courses," included nine ideas under the heading "Earth materials." Six of those papers involved classification (Thomas and Thomas, 2000; Dowse, 2000; Reynolds and Semken, 2000; Harper, 2000; Christman, 2000; Niemitz, 2000). Classification of rocks and minerals is obviously a necessary component of introductory geology courses.

Classification of rocks and minerals is taught in geology courses in order to understand geology better and to provide a basis for communication. Classification of rocks and minerals also provides a rich illustration of mathematical concepts involved in classification in general. These concepts, which are staples of courses in logic, occur in the mathematics curriculum in courses on discrete mathematics.

This column is the second on the connection between sets and geological terminology. The context for the first (CG-18) was planets and moons. The context for this one is rocks.

Getting Started

As discussed in CG-18, a set can be defined either by listing all of its members or by stating a requisite property of its members. In the latter method, the domain of discourse (universal set) must also be stated.

In the examples considered in this column, the domain of discourse will be *rocks*. Thus, to start, we should define the set, *R*, rocks. Here are definitions from two standard textbooks:

> **Rock**. Any naturally formed, nonliving, firm, and coherent aggregate mass of mineral matter that constitutes part of a planet. (Skinner and Porter, 2000, p. G10)
> **Rock**. (1) A solid aggregate of mineral grains. (2) A solid, naturally occurring mass of matter composed of mineral grains, glass, altered organic matter, or combinations of these components. (Raymond, 2002, p. 703)

The second definition from Raymond lends itself to an easy symbolic statement. Let M be the set, "masses of matter." In addition, define the following predicate functions:

$S(x)$ = "x is solid";
$N(x)$ = "x is naturally occurring";
$C_m(x)$ = "x contains some mineral grains";
$C_g(x)$ = "x contains some glass";
$C_o(x)$ = "x contains some altered organic matter."

Then, we can write,

$$R_R = \{x \in M \mid S(x) \wedge N(x) \wedge [C_m(x) \vee C_g(x) \vee C_o(x)]\} \quad (1)$$

for rocks according to Raymond's second definition. ("$\wedge$" is the conjunction, "and"; "$\vee$" is the inclusive disjunction, "or". See CG-10.)

The criterion "solid," however, is not sufficient to exclude masses of unconsolidated materials such as sediment and soils from rocks. Skinner and Porter's "firm and coherent" would serve this purpose, and so we can easily add $F(x)$ for

$F(x)$ = "x is firm and coherent."

to Equation 1. Nor does Equation (1) rule out animals and plants. Therefore, following Skinner and Porter, we can require "nonliving." Thus, let

$L(x)$ = "x is living."

Then, we can define rocks as:

$$R = \{x \in M \mid S(x) \wedge F(x) \wedge \sim L(x) \wedge N(x) \wedge [C_m(x) \vee C_g(x) \vee C_o(x)]\} \quad (2)$$

("~" is the negation, "not". See CG-10.)

Equation 2 is our definition of *rock*.

Subsets and Complements

The *complement* of a set consists of all the elements of the domain that are not members of the set. For example, consider the set R_I defined by

$$R_I = \{x \in R \mid x \text{ formed from a magma}\}, \quad (3)$$

the set of all igneous rocks. R_I . The complement, $R_I^{\ C}$, is

$$R_I^{\ C} = \{x \in R \mid x \notin R_I\} \ , \quad (4)$$

the set of all rocks that are not igneous rocks. The two sets R_I and $R_I^{\ C}$ are illustrated in Figure 1. R_I includes granite, basalt, anorthosite, and many other rocks. $R_I^{\ C}$ includes limestone, sandstone, slate, and many other rocks.

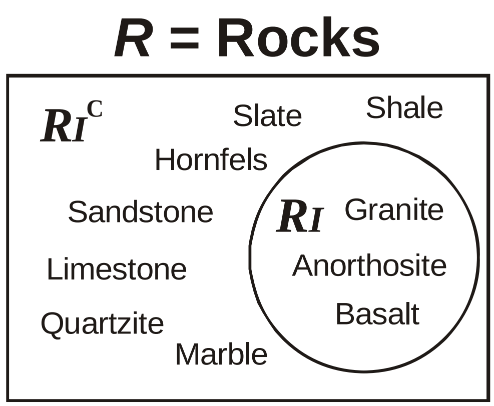

Figure 1. The set of igneous rocks, R_I, including three of its members, and the complement of igneous rocks, $R_I^{\ C}$, including seven of its members.

A *subset* of R consists only of elements of R. Therefore, R_I and $R_I^{\ C}$ are both subsets of R. This is written

$$R_I \subseteq R \qquad R_I^{\ C} \subseteq R. \tag{5}$$

Symbolically, the subset criterion is

$$(R_I \subseteq R) \leftrightarrow [\forall x\, [(x \in R_I) \rightarrow (x \in R)] \tag{6}$$

("$\leftrightarrow$" is the biconditional, if and only if; "$\forall$" is the quantifier, "all"; "$\rightarrow$" is the conditional, "if … then".)

R_I is a *proper subset* of R if all of the elements of R_I are elements of R *and* some elements of R are not included in R_I. Therefore, R_I and $R_I^{\ C}$ also are proper subsets of R. This is written

$$R_I \subset R \qquad R_I^{\ C} \subset R. \tag{7}$$

Symbolically, the criterion for proper subset is

$$(R_I \subset R) \leftrightarrow [\forall x\, [(x \in R_I) \rightarrow (x \in R)] \wedge \exists x\, [(x \in R) \wedge (x \notin R_I)]] \,. \tag{8}$$

("$\exists$" is the quantifier, "some".)

As the symbols suggest, the difference between "subset" and "proper subset" is that the former admits the possibility that $R_I = R$. Thus "$\subseteq$" says "lies totally within or is equal to," whereas "$\subset$" says "lies totally within." More explicitly

$$(R_I \subseteq R) \leftrightarrow [(R_I \subset R) \vee (R_I = R)] \,. \tag{9}$$

The null set (∅) and the domain are both considered subsets of the domain. To illustrate how this works, consider the subset of R_I

$$R_{I,Ol,Qtz} = \{x \in R_I \mid x \text{ contains olivine and quartz}\} \quad (10)$$

the set of all igneous rocks that contain both olivine and quartz. Absent any specialized knowledge, we can say

$$R_{I,Ol,Qtz} \subseteq R_I \qquad R_{I,Ol,Qtz}{}^C \subseteq R_I \quad . \quad (11)$$

But, as geologists, we also know (from Bowen's reaction series) that olivine and quartz will not occur together in igneous rocks. Therefore, we know that

$$R_{I,Ol,Qtz} = \emptyset \qquad R_{I,Ol,Qtz}{}^C = R_I \quad (12)$$

(i.e., $R_{I,Ol,Qtz}$ consists of exactly the same elements as ∅, and $R_{I,Ol,Qtz}{}^C$ consists of exactly the same elements as R_I). By substituting Equations 12 into Equations 11, we have it that ∅ is a subset of R_I, and R_I is a subset of R_I.

This example also illustrates that calling a set a proper subset requires more information than calling it a subset.

I am belaboring the distinction between "subset" and "proper subset" because many geologists instinctively think "subset" means "totally contained within."

Combination of Sets

Two sets can be combined in a variety of ways. To illustrate this, we will use two sets within R (Table 1):

$$R_{Perm} = \{x \in R \mid x \text{ was formed during the Permian Period.}\}, \quad (13)$$

the set of all rocks of Permian age, and

$$R_{Ariz} = \{x \in R \mid x \text{ is located in Arizona.}\}, \quad (14)$$

the set of all rocks in Arizona. R_{Perm} contains the Kaibab Limestone of Arizona, the Phosphoria Formation of Wyoming, the Capitan Limestone of West Texas, and many others. R_{Ariz} contains the Kaibab Limestone of Arizona, the Tapeats Sandstone (Cambrian) of Arizona, and the Vishnu Schist (Precambrian) of Arizona, and many other rocks.

Intersection. Given two sets, S_1 and S_2, their *intersection* $S_1 \cap S_2$ is the set consisting of all the elements that are in both S_1 and S_2. Thus, the intersection of R_{Perm} and R_{Ariz} (Fig. 2A) is

$$R_{Perm} \cap R_{Ariz} = \{x \in R \mid (x \in R_{Perm}) \wedge (x \in R_{Ariz})\}. \quad (15)$$

It consists of all the Permian rocks in Arizona (including only the Kaibab Limestone among the five examples in Table 1).

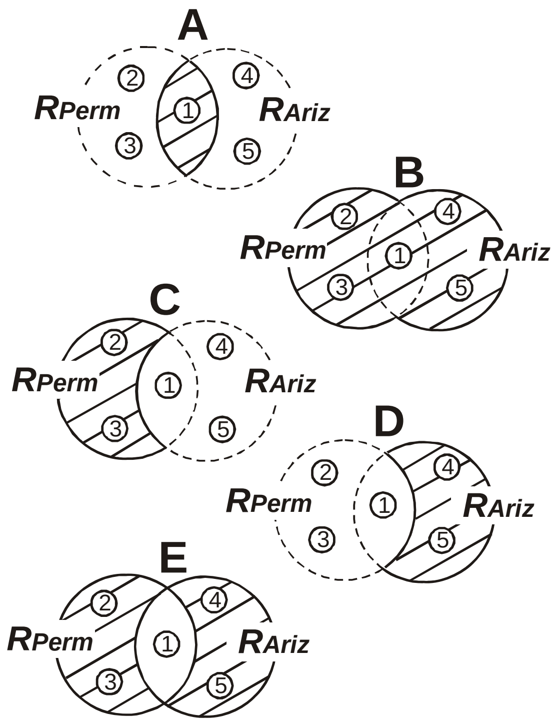

Figure 2. Intersection (A), union (B), differences (C and D), and symmetric difference of the set of rocks of Permian age (R_{Perm}) and the set of rocks located in Arizona (R_{Ariz}). Noted members: (1), Kaibab Ls; (2) Phosphoria Fm; (3) Capitan Ls; (4) Tapeats Ss; (5) Vishnu Schist.

	R_{Perm}	R_{Ariz}	$R_{Perm} \cap R_{Ariz}$	$R_{Perm} \cup R_{Ariz}$	$R_{Perm} - R_{Ariz}$	$R_{Ariz} - R_{Perm}$	$R_{Perm}\ \Delta\ R_{Ariz}$
Kaibab Ls	×	×	×	×			
Phosphoria Fm	×			×	×		×
Capitan Ls	×			×	×		×
Tapeats Ss		×		×		×	×
Vishnu Schist		×		×		×	×

Table 1. Some elements of R_{Perm} and R_{Ariz} and their combinations.

Union. Given two sets, S_1 and S_2, their *union* $S_1 \cup S_2$ is the set consisting of all the elements that are in either S_1 or S_2, or in both S_1 and S_2. Thus, the union of R_{Perm} and R_{Ariz} (Fig. 2B) is

$$R_{Perm} \cup R_{Ariz} = \{x \in R \mid (x \in R_{Perm}) \vee (x \in R_{Ariz})\} \quad . \qquad (16)$$

It consists of all the rocks of Permian age all over the world *plus* all the rocks in Arizona of any age (including all five examples of Table 1).

Difference. Given two sets, their difference $S_1 – S_2$ is the set consisting of all the elements that are in S_1 but not in S_2. Thus, for any two sets, such as R_{Perm} and R_{Ariz}, we have two differences:

$$R_{Perm} - R_{Ariz} = \{x \in R \mid (x \in R_{Perm}) \wedge (x \notin R_{Ariz})\} \tag{17}$$

$$R_{Ariz} - R_{Perm} = \{x \in R \mid (x \in R_{Ariz}) \wedge (x \notin R_{Perm})\} \tag{18}$$

$R_{Perm} - R_{Ariz}$ (Fig. 2C) consists of all the Permian rocks of the world *except* those in Arizona (including the Phosphoria and the Capitan of Table 1). $R_{Ariz} - R_{Perm}$ (Fig. 2D) consists of all the rocks in Arizona *except* those of Permian age (including the Tapeats Sandstone and the Vishnu Schist of Table 1).

From these definitions, the complement of a set can be viewed as the domain less the set. Thus, for R,

$$R^C = M - R \tag{19}$$

Also, using R_{Perm} and R_{Ariz}, there is the following relationship between difference and union that follows from the definitions (Equations 16 and 17):

$$(R_{Perm} - R_{Ariz}) \cup R_{Ariz} = R_{Perm} \cup R_{Ariz} \tag{20}$$

(compare with Figs. 2B and 2C). Moreover, again using R_{Perm} and R_{Ariz} and basic definitions (Equations 15 and 17), the difference can be written in terms of the intersection:

$$R_{Perm} - R_{Ariz} = R_{Perm} - (R_{Perm} \cap R_{Ariz}) \tag{21}$$

(compare with Figs. 2A and 2C).

Symmetric difference. Given two sets, their symmetric difference $S_1 \Delta S_2$ is the set consisting of all the elements that are in S_1 or S_2 but not in both (Fig. 2E). Thus, using R_{Perm} and R_{Ariz} again,

$$R_{Ariz} \Delta R_{Perm} = \{x \in R \mid [(x \in R_{Ariz}) \vee (x \in R_{Perm})] \wedge \sim [(x \in R_{Ariz}) \wedge (x \in R_{Perm})]\} \tag{22}$$

that is, all the Permian rocks in the world and all the rocks in Arizona except the Permian rocks in Arizona. (This combination includes the five examples of Table 1 less the Kaibab Limestone.) An alternative way of writing Equation 22 is

$$R_{Ariz} \Delta R_{Perm} = \{x \in R \mid [x \in (R_{Ariz} \cup R_{Perm})] \wedge [x \notin (R_{Ariz} \cap R_{Perm})]\}. \tag{23}$$

Thus,

$$R_{Ariz}\,\Delta\,R_{Perm} = (R_{Ariz} \cup R_{Perm}) - (R_{Ariz} \cap R_{Perm}). \qquad (24)$$

Moreover, the symmetric difference is the union of the two (asymmetric) differences:

$$R_{Ariz}\,\Delta\,R_{Perm} = (R_{Ariz} - R_{Perm}) \cup (R_{Perm} - R_{Ariz}). \qquad (25)$$

Partitions

Ideally, when one sets up a classification system the domain is already known, and one can proceed to partition it into non-overlapping classes. This can be called *classification by subdivision*.

Often in geology, however, the domain is being discovered as the terminology is being developed. Classes are named as they are encountered. This can be called *classification by sorting*. It is much like separating objects into different piles as one draws them out of a grab bag not knowing what kinds of objects are in the bag.

Classification by sorting typically partitions the domain in a different way than one might think.

How Two Definitions Can Produce Four Classes. Suppose, for example, you make two explicit definitions for classes (sets) within a larger family of objects (domain). This can easily produce four classes, rather than just the two stipulated by the definitions. Thus the definitions of Equations 13 (R_{Perm}) and 14 (R_{Ariz}) actually subdivide R into the following four separate classes (Fig. 3):

1. $R_{Perm} - R_{Ariz}$. Permian rocks that are not in Arizona.
2. $R_{Perm} \cap R_{Ariz}$. Permian rocks in Arizona.
3. $R_{Ariz} - R_{Perm}$. Non-Permian rocks in Arizona.
4. $R - R_{Perm} - R_{Ariz}$. Non-Permian rocks that are not in Arizona.

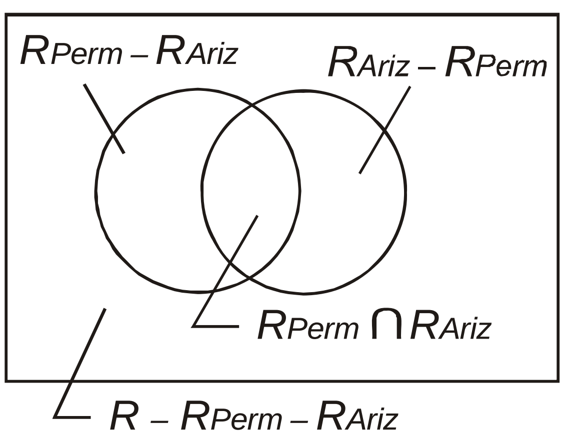

Figure 3. Definition of R_{Perm} and R_{Ariz} partition rocks into four sets: the intersection, the two differences, and the complement of the union.

We get the four classes out of two definitions, of course, because (1) R_{Perm} and R_{Ariz} overlap, and (2) R_{Perm} and R_{Ariz} do not cover all of the possibilities. In other words: (1) the two defined sets are *not mutually exclusive*, and (2) they are *not exhaustive* of R Two defined classes are mutually exclusive if their intersection is null. Defined classes are exhaustive if their union equals the domain. In the language of sets, mutually exclusive sets that exhaust the domain compose a *partition*. A partition (Fig 4) can be pictured as a patchwork quilt (Johnson, 1998, p. 67).

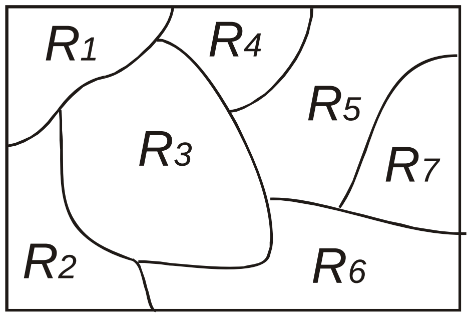

Figure 4. A partition of R into six sets.

Mutually Exclusive. Sets that are mutually exclusive are said to be *disjoint*. For example, instead of R_{Perm} and R_{Ariz}, consider R_{Perm} and R_{AOB}, where

$$R_{AOB} = \{x \in R \mid x \text{ is in the Atlantic Ocean Basin.}\} \qquad (26)$$

Absent geologic considerations, we would draw the four possible sets resulting from the definitions of R_{Perm} and R_{AOB} (Fig. 5A) as we did for the definitions of R_{Perm} and R_{Ariz} (Fig. 3). Before plate tectonics, we would have thought that Figure 5A is the correct way to represent R_{Perm} and R_{AOB}. That is, we would have known that there are rocks that are neither Permian nor located in the Atlantic Ocean Basin (the complement of the union), and, although no one had ever found them, we would have thought that there are Permian rocks in the Atlantic Ocean Basin (the intersection).

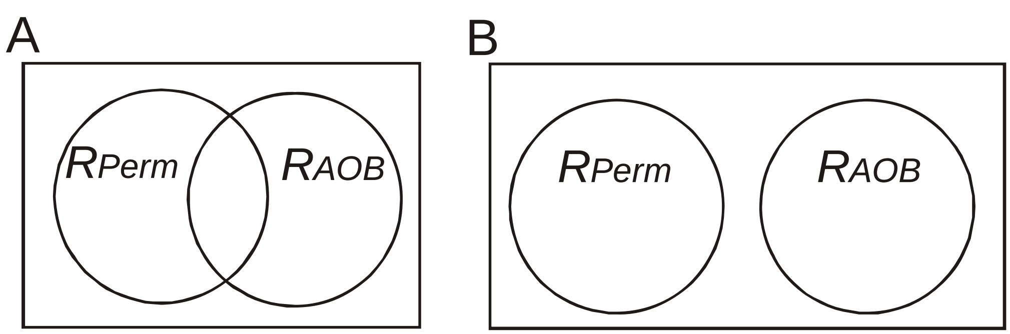

Figure 5. The intersection of R_{Perm} and the set of rocks located in the Atlantic Ocean Basin (R_{AOB}) is null; R_{Perm} and R_{AOB} are disjoint.

According to plate tectonics, however, the intersection $R_{Perm} \cap R_{AOB}$ is null. R_{Perm} and R_{AOB} are disjoint. Therefore, we can draw them as in Figure 5B. If it ever turns out that $R_{Perm} \cap R_{AOB}$ is not null, then it will be back to the drawing board for plate tectonics.

Making a partition. Four sets R_1, R_2, R_3 and R_4 form a partition of R if and only if they are exhaustive and mutually exclusive. The straightforward way of designing a partition – and geologists try to do this all the time – is to subdivide the domain by segmenting one or more scales. One obvious example for R is to divide it according to age. For example: the following four sets provide a partition of R:

$R_{PreC} = \{x \in R \mid x \text{ is Precambrian in age}\}$
$R_{Paleoz} = \{x \in R \mid x \text{ is Paleozoic in age}\}$
$R_{Mesoz} = \{x \in R \mid x \text{ is Mesozoic in age}\}$
$R_{Cenoz} = \{x \in R \mid x \text{ is Cenozoic in age}\}$

They provide a partition because it is impossible for a single rock to have more than one age of origin (mutually exclusive) and the entire geologic time scale is covered (exhaustive).

Defining a set immediately establishes a partition of the domain into the set and its complement. For example, R_I (igneous rocks) and R_I^C (non-igneous rocks) (Fig. 1) form a partition of R. R_{PreC} (Precambrian rocks) and R_{PreC}^C (Phanerozoic rocks) also form a partition of R. The subdivision of rocks by origin (R_I and R_I^C) is completely independent of the subdivision by age (R_{PreC} and R_{PreC}^C). The two subdivisions, therefore, occur along separate dimensions. They can be used as classes on the margins (row labels and column labels) of a bilateral classification scheme (Fig. 6). The four intersections of rows and columns form a partition of R: Phanerozoic igneous rocks (upper left cell of Fig. 6); Phanerozoic non-igneous rocks (upper right); Precambrian igneous rocks (lower left); Precambrian non-igneous rocks (lower right).

	R_I	R_I^C
R_{PreC}^C	$R_{PreC}^C \cap R_I$	$R_{PreC}^C \cap R_I^C$
R_{PreC}	$R_{PreC} \cap R_I$	$R_{PreC} \cap R_I^C$

Figure 6. Igneous and non-igneous rocks partition rocks (columns), as do Precambrian and Phanerozoic rocks (rows). The cells formed by the intersections of the rows and columns form a partition.

R_{PreC}^C is the same as the union of R_{Paleoz}, R_{Mesoz}, and R_{Cenoz}. In contrast, R_I^C is not the same as the union of sedimentary rocks and metamorphic rocks, because igneous rocks, sedimentary rocks, and metamorphic rocks do not partition rocks (next column).

De Morgan's Laws of Logic

George Boole (1815-1864), the English logician who created the algebra of logic and sets, was a self-taught mathematician. Hollingdale (1989) gives an account of how Boole, needing to support his parents, became an elementary school teacher at age 16 after teaching himself Latin and Greek. At 19 (the time of Lyell's *Principles of Geology*

[1830], and while Darwin was aboard the *Beagle* [1831-1836]), young George opened a school of his own. Then, according to Hollendale (1989, p. 342):

> The need to teach his pupils mathematics aroused his interest in the subject; his innate ability enabled him to read and to understand, entirely on his own, some of the most difficult works of such masters as Laplace and Lagrange. It was not long before this remarkable young man was making discoveries of his own, mainly in the field of what we now call 'abstract algebra'.

Boole's seminal treatise (1854, five years before Darwin's *Origin*) was *An Investigation of the Laws of Thought, on Which are Founded the Mathematical Theories of Logic and Probabilities*. The work was continued after his death by his friend Augustus De Morgan (1806-1871), the immensely influential Professor of Mathematics at University College London; Benjamin Peirce (1809-1880), the Harvard mathematician and astronomer who determined the orbit and perturbations of the newly discovered Neptune; and Peirce's son, Charles Sanders Peirce (1839-1914), the American geophysicist, logician, and philosopher (see CG-18).

De Morgan was one of the most highly regarded mathematicians in Britain, partly for his innovative research in logic, but mainly because he was a great mathematics teacher. According to Rice (1999),

> (At University College), he single-handedly delivered courses on mathematics to a generation of undergraduates for a third of a century. Because he was in charge of mathematical tuition at the leading higher educational institutional in his nation's capital, he was a formative influence on numerous mathematicians, scientists, and other prominent intellectual figures of the Victorian period.

De Morgan lectured mornings and afternoons, six days a week, and wrote hundreds of notebooks that he placed in the University College library to supplement his oral instruction. This tireless and dedicated professor, who twice resigned his position on matters of principle, was remembered fondly by his students (Rice, 1999).

De Morgan and Benjamin Peirce discovered the "laws of duality" (Hollingdale, 1989, p. 348) that relate negations, conjunctions and disjunctions in the logic of propositions, and complements, unions, and intersections in the algebra of sets. These exceedingly useful laws are known as De Morgan's laws to all students of logic and discrete mathematics.

Let p and q be propositions (statements that can be T or F, see CG-10). Then, the two De Morgan Laws for propositions are

$$\sim(p \wedge q) \leftrightarrow \sim p \vee \sim q \tag{27a}$$
$$\sim(p \vee q) \leftrightarrow \sim p \wedge \sim q \quad . \tag{27b}$$

The first statement (Equation 27a) says that the negation of a conjunction is logically equivalent to the negations of the individual disjuncts. The second statement (Equation 27b) says that the negation of a disjunction is logically equivalent to the negations of the individual conjuncts. Both laws are easily proved by means of truth tables (CG-10).

Now, let $P(x)$ and $Q(x)$ be predicate functions. De Morgan's Laws become

$$\sim[P(x) \wedge Q(x)] \leftrightarrow \sim P(x) \vee \sim Q(x) \quad (28a)$$
$$\sim[P(x) \vee Q(x)] \leftrightarrow \sim P(x) \wedge \sim Q(x) \quad . \quad (28b)$$

As an example, let x refer to igneous rocks, and define the functions P and Q as

$P(x)$ = x contains plagioclase. (29a)
$Q(x)$ = x contains olivine. (29b)

Then, the left-to-right part of Equation 28a says: "If it is not the case that the rock (x) contains both plagioclase and olivine, then it lacks one or both of the two minerals. The right-to-left part of the first law says: "If the rock does not contain plagioclase and/or does not contain olivine, then its does not contain both minerals." The left-to-right- part of Equation 28b says: "If it is not the case that the rock contains either or both plagioclase and olivine, then it does not contain plagioclase and it does not contain olivine." The right-to-left part says: "If the rock lacks plagioclase and lacks olivine, then it cannot be said to contain plagioclase and/or olivine."

In addition to applying the laws (Equations 27) to two predicate functions (Equations 28), we can apply the laws to two instances of one predicate function. For example, let x be a crystal in a rock consisting of exactly two crystals (x_1 and x_2), and define the function Q as

$Q(x)$ = x is olivine. (30)

Then, applying De Morgan's Laws,

$$\sim[Q(x_1) \wedge Q(x_2)] \leftrightarrow \sim Q(x_1) \vee \sim Q(x_2) \quad (31a)$$
$$\sim[Q(x_1) \vee Q(x_2)] \leftrightarrow \sim Q(x_1) \wedge \sim Q(x_2). \quad (31b)$$

According to Equation 31a, "It is not the case that both crystals are olivine" is logically equivalent to "either the first crystal is not olivine, or the second is not olivine, or they both are not olivine." According to Equation 31b, "It is not the case that one or the other or both of the crystals are olivine" is logically equivalent to "The first crystal is not olivine and the second crystal is not olivine."

Equations 31 can be generalized to a rock consisting of n crystals. Thus:

$$\sim[Q(x_1) \wedge Q(x_2) \wedge \ldots \wedge Q(x_n)] \leftrightarrow \sim Q(x_1) \vee \sim Q(x_2) \vee \ldots \vee \sim Q(x_n) \quad (32a)$$
$$\sim[Q(x_1) \vee Q(x_2) \vee \ldots \vee Q(x_n)] \leftrightarrow \sim Q(x_1) \wedge \sim Q(x_2) \wedge \ldots \wedge \sim Q(x_n) \quad . \quad (32b)$$

The left-hand side of Equation 32a says "it is not the case that all of the crystals are olivine." The right-hand side of the Equation 32a says, "at least one of the crystals is not olivine," (or "some of the crystals are not olivine"). The left-hand side of Equation 32b says, "it is not the case that at least one of the crystals is olivine.” The right-hand side of Equation 32b says, "none of the crystals is olivine" (or "each of the crystals is not olivine," or "every one of the crystals is not olivine.")

We can use universal and existential quantifiers (CG-18) for statements involving "all of the crystals" and "at least one of the crystals," respectively. Thus Equations 32 can be rewritten as

$$\sim[\forall x\ Q(x)] \leftrightarrow [\exists x\ \sim Q(x)] \quad (33a)$$
$$\sim[\exists x\ Q(x)] \leftrightarrow [\forall x\ \sim Q(x)] \quad (33b)$$

The first says "*not all are* is logically the same as *some are not*." The second says "*not some are* is logically the same as *all are not*."

Textural Terms for Igneous Rocks. The textural terms *aphanitic* and *phaneritic* are familiar to everyone who has taken introductory physical geology. The terms, however, are used inconsistently in standard textbooks. Depending on how they are defined, they may or may not partition the set of igneous rocks. Confusion on this point illustrates the important logical principle that *opposite is not the same as negation*.

One does not have to search far to find the logical inconsistency. The first three books I picked up to find definitions were Raymond (2002), Skinner and Porter (2000) and Ehlers and Blatt (1982).

Raymond has the following definitions in his glossary:

> **Aphanitic**. A descriptive term meaning that the grains in a rock are too small to see or identify with either the unaided eye or a low-power lens.
> **Phaneritic**. A descriptive term applied to crystalline materials in which grains can be discerned without the aid of a microscope.

Skinner and Porter do not include the terms in their glossary, but they do have aphanite and phanerite:

> **Aphanite**. An igneous rock in which the constituent mineral grains are so small they can only be seen clearly by using some kind of magnification.
> **Phanerite**. An igneous rock in which the constituent mineral grains are readily visible to the unaided eye.

Ehlers and Blatt (1982) do not have a glossary, but say the following about the two textural terms in the context of features of igneous rocks to be seen in hand samples:

> The first distinction to be made is whether or not individual mineral grains can be seen. If the mineral grains can be seen the rock is classified as *phaneritic*; if not, it is *aphanitic*.

One of the areas of inconsistency in these definitions is the kind of rocks the words are applied to. By tradition, the words apply to igneous rocks. By Raymond's definitions, aphanitic can be applied to rocks in general, and phaneritic can be applied to crystalline (as opposed to granular) rocks. Thus, according to these definitions, aphanitic rocks could include shale, fine-grained limestone, chert, and slate, as well as basalt, and phaneritic rocks could include coarse marble, as well as granite. This is not the inconsistency I am interested in, however.

I am interested in the large differences in meanings implied by seemingly small differences in wording, such as the presence and absence of the word *the* (as in "the grains" in Raymond's definition of aphanitic). To see the logical inconsistencies that result from such differences, we can write the defining criteria symbolically. Thus, let y refer to constituent grains, and define the predicate function $V(y)$ by

$$V(y) = y \text{ can be seen without a hand lens.} \quad (34)$$

Then the various criteria are:

- Aphanitic according to Raymond and to Skinner and Porter: *the constituent grains are not visible*,
  $$\forall y \sim V(y). \quad (35)$$
- Aphanitic according to Ehlers and Blatt: *not all of the constituent grains are visible*,
  $$\sim[\forall y\ V(y)]. \quad (36)$$
- Phaneritic according to Skinner and Porter and to Ehlers and Blatt: *the constituent grains are visible*,
  $$\forall y\ V(y). \quad (37)$$
- Phaneritic according to Raymond: *some constituent grains are visible*,
  $$\exists y\ V(y). \quad (38)$$

How do these criteria compare to each other?

Clearly Statements 36 and 37 are direct negations of each other. Applying the second De Morgan law (Equation 33b) to Statement 35 produces

$$\forall y \sim V(y) = \sim[\exists y \sim\sim V(y)],$$

which, after applying the Law of Double Negation to the right side, becomes

$$\forall y \sim V(y) = \sim[\exists y\ V(y)]. \quad (39)$$

Thus by Equation 39, Statements 35 and 38 are also negations of each other. Meanwhile, the first De Morgan law (Equation 33a) applied to Statement 36 produces

$$\sim[\forall y\ V(y)] = \exists y \sim V(y), \quad (40)$$

which shows that Statement 36 can be stated alternatively as "some constituent grains are not visible."

The four statements (35-38), therefore, differ in a very classic way. Statement 37 is the universal affirmative proposition ("all grains are visible" in this example). Statement 35 is the universal negative proposition ("no grains are visible"). Statement 38 is the existential affirmative proposition ("some grains are visible"). Statement 36 is the existential negative ("some grains are not visible"). These four propositions (labeled A, E, I, and O, respectively, in classic categorical logic) are at the corners of the famous square of opposition (Fig. 7A) dating back to Aristotle.

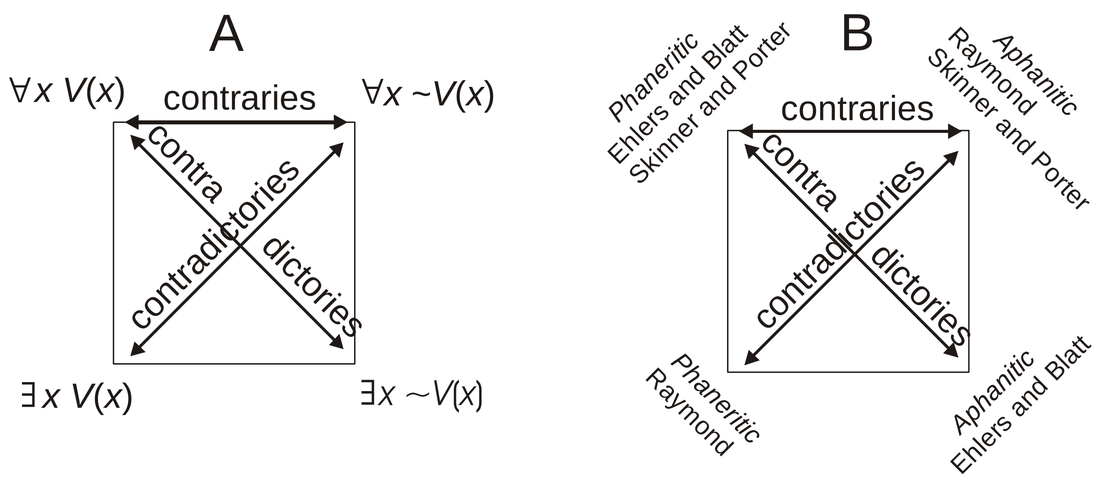

Figure 7. Square of opposition using the classic categorical propositions (A) and illustrated by the three different definitions of aphanitic and phaneritic (B).

In the traditional language of categorical propositions, the propositions across the diagonals of the square of opposition are contradictories. Universal propositions at horizontally opposite corners are contraries.

Thus (Fig. 7B) Raymond's aphanitic and phaneritic are contradictories (i.e., negations). Ehlers and Blatt's aphanitic and phaneritic are also contradictories (negations), but in a different way. Skinner and Porter's aphanitic and phaneritic are not contradictories, but rather contraries (opposites). It is in this way that usage of the terms aphanitic and phaneritic illustrate the confusion between negations ("nots") and opposites.

Textural Partition of Igneous Rocks. Contradictories (negations) can be used to partition a domain into two sets. If there are more than two elements in the domain, contraries (opposites) will not partition the domain. We can illustrate this concept by making sets from the various definitions of aphanitic and phaneritic.

First, let $C(x,y)$ be the predicate function,

$$C(x,y) = y \text{ is contained in a hand sample of } x. \qquad (41)$$

Then we can define the following four sets.

- The set of igneous rocks in which *the constituent grains are not visible*:

$$R_{I,A1} = \{x \in R_I \mid \forall y\, [C(x,y) \rightarrow \sim V(y)]\}. \qquad (42)$$

This is the first definition of aphanitic igneous rocks (aphanites) -- using the criterion of Raymond and of Skinner and Porter.

- The set of igneous rocks in which *not all of the constituent grains are visible*:

$$R_{I,A2} = \{x \in R_I \mid \exists y\, [C(x,y) \wedge \sim V(y)]\}. \quad (43)$$

This is the second definition of aphanitic ignous rocks -- using the criterion of Ehlers and Blatt.

- The set of igneous rocks in which *the constituent grains are visible*:

$$R_{I,P1} = \{x \in R_I \mid \forall y\, [C(x,y) \rightarrow V(y)]\} \quad (44)$$

This is the first definition of phaneritic igneous rocks (phanerites) -- using the criterion of Skinner and Porter and of Ehlers and Blatt.

- The set of igneous rocks in which *some constituent grains are visible*:

$$R_{I,P2} = \{x \in R_I \mid \exists y\, [C(x,y) \wedge V(y)]\} \quad (45)$$

This is the second definition of phaneritic igneous rocks -- using the criterion of Raymond.

These sets are shown on Figure 8. Raymond's pair of definitions partitions the igneous rocks one way, and Ehler and Blatt's pair of definitions partitions igneous rocks the other way. Skinner and Porter's pair of definitions (all grains not visible and all grains visible) does not partition igneous rocks. It leaves unnamed the set of porphyritic igneous rocks with aphanitic groundmass:

$$R_3 = \{x \in R_I \mid [\exists y\, [C(x,y) \wedge V(y)]] \wedge [\exists y\, [C(x,y) \wedge \sim V(y)]]\}, \quad (46)$$

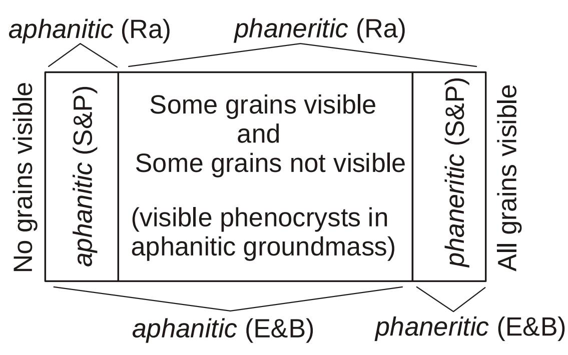

Figure 8. How the three different definitions of aphanitic and phaneritic igneous rocks partition, or fail to partition, the domain. Ra = Raymond (2002); S&P = Skinner and Porter (2000); E&B = Ehlers and Blatt (1982).

igneous rocks that contain some grains that are visible and some grains that are not visible.

To summarize,

- With Raymond's definitions,
  $R_{I,A1} \cup R_{I,P2} = R_I$, and $R_3 \subseteq R_{I,P2}$

- With Ehlers and Blatt's definitions,
  $R_{I,A2} \cup R_{I,P1} = R_I$, and $R_3 \subseteq R_{I,A2}$
- With Skinner and Porter's definitions,
  $R_{I,A1} \cup R_{I,P1} \cup R_3 = R_I$.

The pervasiveness of the conflation of negations and opposites can be illustrated within a single source. In the quotation from Ehlers and Blatt discussed so far, aphanitic and phaneritic are negations (contradictories) that partition igneous rocks. In a different part of the book (p. 102), where Ehlers and Blatt discuss the classification of igneous rocks, they refer to phaneritic rocks "whose grains are sufficiently coarse to be individually distinguishable" and aphanitic rocks "whose grains are too small to be individually distinguishable." This is the distinction made by Skinner and Porter. It treats aphanitic and phaneritic as opposites (contraries). These criteria will not partition igneous rocks.

De Morgan's Laws for Sets

De Morgan's Laws of Logic can be used with the definitions of complement, union, and intersection to derive useful relationships between combinations of sets.
De Morgan's first law for two sets says that the complement of the intersection is the same as the union of the individual complements:

$$(S_1 \cap S_2)^C = S_1^C \cup S_2^C \quad . \qquad (47a)$$

De Morgan's second law for two sets says that the complement of the union is the same as the intersection of the individual complements:

$$(S_1 \cup S_2)^C = S_1^C \cap S_2^C \quad . \qquad (47b)$$

Equation 47a corresponds to Equation 27a, about the negation of a conjunction. Equation 47b corresponds to Equation 27b, about the negation of a disjunction. The laws for two sets can be generalized to treat more than two sets:

$$\left(\bigcap_{i=1}^{n} S_i \right)^C = \bigcup_{i=1}^{n} S_i^C \qquad (47c)$$

$$\left(\bigcup_{i=1}^{n} S_i\right)^C = \bigcap_{i=1}^{n} S_i^C \qquad (47d)$$

We can explore De Morgan's laws for sets by examining combinations of the set of igneous rocks that contain plagioclase and the set of igneous rocks that contain alkali feldspar (Kspar and albite). In order to conform to the classification of the International Union of Geological Sciences (IUGG; see Raymond, 2002, Figs. 3.7, 3.11 and Ehlers and Blatt, Fig. 4-1), we need to use precise (though long) defining criteria. Thus for rocks that we will refer to simply as "igneous rocks that contain plagioclase", let

$R_{I,Plag}$ = {$x \in R_I$ | >10% by volume of *x* is feldspar and >10% of the feldspar in *x* is plagioclase with Ca content >An05}. (48a)

For rocks that we will refer to as "igneous rocks that contain alkali feldspar," let

$R_{I,Alkspar}$ = {$x \in R_I$ | >10% by volume of *x* is feldspar and >10% of the feldspar in *x* is Kspar and/or plagioclase with Ca content ≤ An05}. (48b)

In order to track the rocks through the various complements and combinations, we will consider ten phaneritic igneous rocks as elements of R_I. Thus

R_I = {alkali feldspar granite, granite, granodiorite, tonalite, monzonite, gabbro, anorthosite, carbonatite, dunite, pyroxenite, …}. (49)

Note the ellipsis (…) in Equation 49. This means that these ten rock types are only selected examples. They are selected because they are in named fields on the IUGS classification diagrams, and they are standard in petrology labs.

So, here we go:

Igneous rocks that contain plagioclase include:

$R_{I,Plag}$ = {granite, granodiorite, tonalite, monzonite, gabbro, anorthosite, …}. (50)

Igneous rocks that contain alkali feldspar include:

$R_{I,Alkspar}$ = {alkali feldspar granite, granite, granodiorite, monzonite, …}. (51)

From Equation 50, igneous rocks that do not contain plagioclase include:

$R_{I,Plag}{}^{C}$ = {alkali feldspar granite, carbonatite, dunite, pyroxenite, …}. (52)

From Equation 51, igneous rocks that do not contain alkali feldspar include:

$$R_{I,Alkspar}^{C} = \{\text{tonalite, gabbro, anorthosite, carbonatite, dunite, pyroxenite, …}\}. \qquad (53)$$

From Equations 52 and 53, igneous rocks that do not contain plagioclase *AND* do not contain alkali spar include:

$$R_{I,Plag}^{C} \cap R_{I,Alkspar}^{C} = \{\text{carbonatite, dunite, pyroxenite, …}\}. \qquad (54)$$

From Equations 52 and 53, igneous rocks that do not contain plagioclase *OR* do not contain alkali spar include:

$$R_{I,Plag}^{C} \cup R_{I,Alkspar}^{C} = \{\text{alkali feldspar granite, tonalite, gabbro, anorthosite, carbonatite, dunite, pyroxenite, …}\}. \qquad (55)$$

From Equation 50 and 51, igneous rocks that contain both feldspars include:

$$R_{I,Plag} \cap R_{I,Alkspar} = \{\text{granite, granodiorite, monzonite, …}\}. \qquad (56)$$

From Equations 50 and 51, igneous rocks that contain at least one feldspar include:

$$R_{I,Plag} \cup R_{I,Alkspar} = \{\text{alkali feldspar granite, granite, granodiorite, tonalite, monzonite, gabbro, anorthosite, …}\}. \qquad (57)$$

From Equation 56, igneous rocks that do not contain both feldspars include:

$$(R_{I,Plag} \cap R_{I,Alkspar})^{C} = \{\text{alkali feldspar granite, tonalite, gabbro, anorthosite, carbonatite, dunite, pyroxenite, …}\}. \qquad (58)$$

From Equation 57, igneous rocks that do not contain either feldspar include:

$$(R_{I,Plag} \cup R_{I,Alkspar})^{C} = \{\text{dunite, carbonatite, pyroxenite, …}\}. \qquad (59)$$

For De Morgan's first law for sets, compare the sets in Equations 55 and 58: the set of igneous rocks that do not contain both feldspars (i.e., for which it is not the case that they contain both feldspars) consists of igneous rocks that (a) do not contain plagioclase and/or (b) do not contain alkali feldspar. For the second law compare the sets in Equations 54 and 59: the set of igneous rocks that do not contain either feldspar consists of igneous rocks that (a) do not contain plagioclase and (b) do not contain alkali feldspar.

Concluding Remark

In the 2001 *Almanac* issue of the *Chronicle of Higher Education*, a table reports that >99% of the faculty at American colleges and universities believe that one of their responsibilities is to encourage students to think. According to Durant (1953, p. 58), "logic means simply the art and method of correct thinking." Boole entitled his masterpiece on logic and sets, *The Laws of Thought*. The connection between terminology, classification, sets, and thinking is clear. The extensive terminology of geology in general, and of rocks in particular, provides a rich vein in which to mine this connection. I can't help but to think, too, that developing this connection would lead to improved communication – and thinking – about geology. Giving students experience in "the art and method of correct thinking" promotes a habit of mind that can be applied far beyond the classroom and far beyond geology.

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