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Computational Geology 15

More Mapping with Vectors

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Topics this issue --

- *Mathematics: general – significant figures; geometry – area of triangles; vectors – addition, cross- product, dot product; algebra – linear equations, Cramer's rule; trigonometry – law of sines, law of cosines; spreadsheets.*
- *Geology: bearing and azimuth; triangulation.*

Introduction

Triangulation is so important that the word is included in general dictionaries:

> "**triangulation** … trigonometric operation for finding a position or location by means of bearings from two fixed points a known distance apart."
>
> *Merriam-Webster's Collegiate Dictionary*, Tenth Edition, 1994.

The essential ingredients are shown in Figure 1. *O* and *P* are the two fixed points a known distance apart. The line *OP* is a baseline. The two bearings are the directions of the sightings from *O* to *A* and from *P* to *A*. The location of *A* is determined as the intersection of lines *OA* and *PA*.

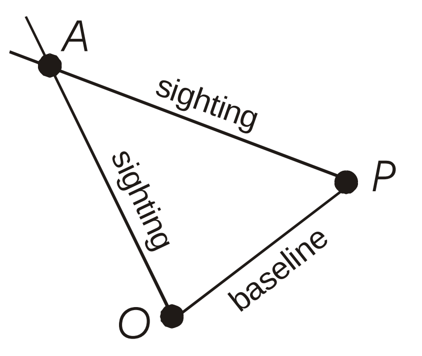

Figure 1. Definition diagram for the essential ingredients of a triangulation problem.

An example is shown in Figure 2. The baseline is 51.2 m long and extends in a direction N65E (*AZ* = 65°, see Appendix). The bearing from *O* to *A* is N10E (*AZ* = 10°), and the bearing from *P* to *A* is N5W (*AZ* = 355°). The question is, Where is *A*?

The answer is commonly obtained graphically by drawing the baseline to scale and using a protractor to place *OA* and *PA*. With the scale drawing of triangle *AOP*, one can measure the distance of *A* from *O* and the distance from *P* to *A*. Thus, in this example, answers to the question are, *A* is 190 m from *O* in a direction N10E and (or) 160 m from *P* in a direction N5W.

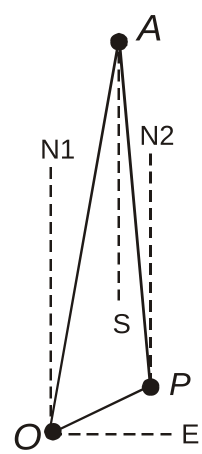

Figure 2. Map showing information for a sample triangulation problem.

The uncertainty in the result depends not only on the accuracy and precision of the original field data but also on the accuracy and precision of the graphical work. The graphically determined result, therefore, is known less well than the original data.

Trigonometric Solution

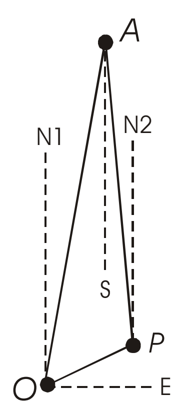

Figure 3. Map showing angles used in the law of sines to find the position of A. E and S indicate directions to the east and to the south, respectively, and N_1 and N_2 indicate the direction to north.

The same results can be calculated using trigonometry as shown in Figure 3. The direction of the baseline *OP* and the two bearings can be used to find all the angles of the triangle. Thus:

- From the fact that *OP* extends in the direction N65E, $\angle POE$ must be 25°. From the fact that *OA* is N10E, $\angle N_1OA$ is 10°. Then, $\angle AOP$ is 55° because $\angle N_1OA$, $\angle AOP$, and $\angle POE$ must sum to 90°.
- From the fact that $\angle N_1OA$ is 10°, $\angle SAO$ also is 10°, because the two angles are formed by the intersection of *OA* with two parallel lines. For the same reason, $\angle PAS$ is 5° because $\angle APN_2$ is 5°. Then, $\angle PAO$ is 15° because it is the sum of $\angle SAO$ and $\angle PAS$.
- From the findings that $\angle AOP$ is 55° and $\angle PAO$ is 15°, then $\angle OPA$ must be 110° because the three angles must sum to 180°.

The angles of the triangle can be combined with the known length (*OP*) to find the two unknown lengths (*OA* and *PA*) by using one of the most helpful relationships in trigonometry, the *law of sines*. According to the law of sines, the ratio of the sine of the vertex angle of a triangle to the length of the opposite side is the same for all three vertices. Thus if α, β, and γ are the angles at *A*, *B*, and *C*, respectively, and if *a*, *b*, and *c* are the lengths of the sides opposite α, β, and γ, respectively, then the law of sines states:

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c} \ . \qquad (1)$$

In the notation of Figure 3, the law of sines is:

$$\frac{\sin\angle PAO}{51.2} = \frac{\sin\angle OPA}{OA} = \frac{\sin\angle AOP}{PA} \, . \qquad (2)$$

Solving for OA and PA,

$$OA = \frac{\sin 110^o}{\sin 15^o} * 51.2 \text{ m} = 185.8915 \text{ m}. \qquad (3a)$$

and

$$PA = \frac{\sin 55^o}{\sin 15^o} * 51.2 \text{ m} = 162.0460 \text{ m}. \qquad (3b)$$

(I am using excess digits in the results in Equations 3, because I intend to do more with the numbers in the next section.)

The calculated results of Equations 3 have no more uncertainty than the uncertainty produced from the uncertainties in the original length (51.2 m) and angles (10° and 5°). Interpreting these data to mean $OP = 51.20 \pm 0.05$ m, $\angle N_1OA = 10.0° \pm 0.5°$, and $\angle APN_2 = 5.0° \pm 0.5°$, and using the rules of error propagation (assuming no partial cancellation; Taylor, 1997), the results are $OA = 185.9 \pm 12.9$ m and $PA = 162.0 \pm 11.7$ m. With this much propagated error, it is inappropriate to state the result with more than two significant digits (i.e., *OA* = 190 m, and *PA* = 160 m), and even that *understates* the uncertainty. Clearly the effects of the uncertainty in the original data are large enough that one would not want to add more uncertainty by using an inexact technique of find *OA* and *PA* – which is the case when one uses graphical techniques where analytical ones are available.

Vectors and Components

From the law of sines and the original data, we know all the sides and all the angles of the triangle *OPA*. This means that we know the location of each vertex relative to the location of each of the other vertices. For example, *P* is 51.2 m in a direction N65E from *O*, and *P* is also 185.8915 m in a direction of S5E (*AZ* = 175°) from *A* (keeping extra digits for future calculations). Similarly, the location of *O* can be stated as a distance and direction from *P* and from *A*, and *A* can be stated as a distance and direction from *O* and from *P* (the latter pair of distances being the solution we obtained in the preceding section).

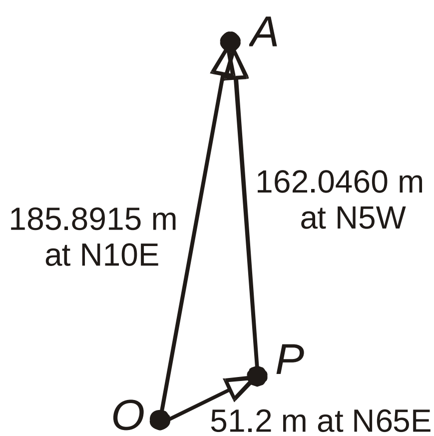

Figure 4. Map showing vectors giving the position of *P* relative to *O*, *A* relative to *O*, and *A* relative to *P*.

These relative locations can be considered as vectors inasmuch as they are directed distances having magnitude (length) and direction. Thus $\mathbf{v}_{O\text{-}P}$, read "vector from O to P," is the quantity that has magnitude 51.2 m and direction N65E. Similarly, $\mathbf{v}_{O\text{-}A}$ has magnitude 185.8915 m and direction N10E, and $\mathbf{v}_{P\text{-}A}$ has magnitude 162.0460 m and direction N5W. These three vectors are shown in Figure 4.

While geologists give directions as azimuths or bearings, the mathematics convention for stating the direction of a vector is to use θ, measured counterclockwise from the *x*-axis (Appendix 1). With this convention, and aligning the east and north directions with the *x*- and *y*-axes, respectively, the three vectors of Figure 4 are:

- $\mathbf{v}_{O\text{-}P}$, with 51.2 m and direction $\theta = 25°$;
- $\mathbf{v}_{O\text{-}A}$, with 185.8915 m and direction $\theta = 80°$;
- $\mathbf{v}_{P\text{-}A}$, with 162.0460 m and direction $\theta = 95°$.

For mathematical manipulations (hence problem solving), it is usually more convenient to express vectors in terms of their components (see Computational Geology-4, Mapping with vectors, Jan. 1999). For the two-dimensional vector $\mathbf{v}$ (Fig. 5)

$$\mathbf{v} = v_x\,\mathbf{i} + v_y\,\mathbf{j}\,, \tag{4}$$

where v_x and v_y are the x-and y-components, respectively, **i** and **j** are the unit vectors in the x- and y-directions, respectively, and the + signifies vector addition (because $v_x\mathbf{i}$ and $v_y\mathbf{j}$ are vectors, each being the scalar component multiplied by a unit vector). As shown in Figure 5, the components, v_x and v_y are related to the magnitude and direction of **v** by:

$$v_x = |\mathbf{v}| \cos\,\theta \tag{5a}$$
$$v_y = |\mathbf{v}| \sin\,\theta \tag{5b}$$

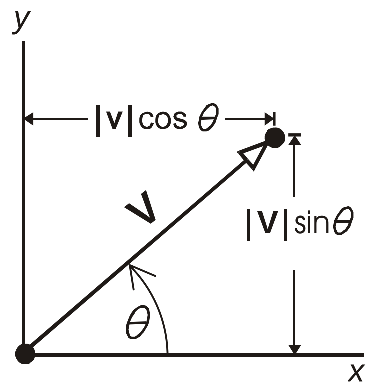

Figure 5. Sketch showing the components of a vector.

Going the other way, the magnitude and direction of **v** can be found from its components by

$$|\mathbf{v}| = \sqrt{v_x^2 + v_y^2} \tag{6a}$$

$$\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right) \tag{6b}$$

One must be careful using the inverse tangent in Equation 6b, because the tangent has a period of only 180° (Appendix).

vector	magnitude (m)	θ (deg)	v_x (m)	v_y (m)
$\mathbf{v}_{\mathbf{O\text{-}P}}$	51.2	25	46.4030	21.6381
$\mathbf{v}_{\mathbf{O\text{-}A}}$	185.8915	80	32.2797	183.0674
$\mathbf{v}_{\mathbf{P\text{-}A}}$	162.0460	95	–14.1232	161.4293

Table 1. Vector components from the law of sines (Fig. 3)

The components of the vectors in Figure 4 are found from Equations 5 and listed in Table 1. With these components, the vectors are:

- $\mathbf{v}_{\mathbf{O\text{-}P}} = 46.4030\ \mathbf{i} + 21.6381\ \mathbf{j}$; (7a)
- $\mathbf{v}_{\mathbf{O\text{-}A}} = 32.2797\ \mathbf{i} + 183.0674\ \mathbf{j}$; (7b)
- $\mathbf{v}_{\mathbf{P\text{-}A}} = -14.1232\ \mathbf{i} + 161.4293\ \mathbf{j}$. (7c)

(Again, the coefficients in Equations 7 contain an absurd number of digits; I am carrying them in order to compare results of different methods.)

In words, the components of Equation 7 say that one can get from *O* to *P* by going 46.4030 m east and then 21.6381 m north. Alternatively, one can say that *P* is 46.4030 m east and 21.6381 m north of *O*. Similarly, *A* is 32.2797 m east and 183.0674 m north of *O*, and *A* is 14.1232 m *west* and 161.4283 m north of *P*.

As shown in Figure 4, one can get from *O* to *A* also by going from *O* to *P*, and then from *P* to *A*. Mathematically, this amounts to adding the vectors,

$$\mathbf{v}_{\mathbf{O\text{-}A}} = \mathbf{v}_{\mathbf{O\text{-}P}} + \mathbf{v}_{\mathbf{P\text{-}A}}\,. \tag{8}$$

Vector addition is easily accomplished by adding the components. Thus,

$$\mathbf{v}_{\mathbf{O\text{-}A}} = (46.4030 - 14.1232)\,\mathbf{i} + (21.6381 + 161.4293)\,\mathbf{j}, \tag{9}$$

which produces Equation 7b.

Coordinates

The position of every point in a mapped area can be stated in terms of its eastward and northward distances (displacements) from any other point in the area. If one of these points is taken as a reference – a local benchmark – then these eastward and northward displacements are the *x*- and *y*-coordinates, respectively, relative to a coordinate origin at that benchmark. For example, if *O* is taken as the local benchmark (i.e., the local origin), then the coordinates of *P* and *A* are $x_P = 46.4030$ and $y_P = 21.6381$, and $x_A = 32.2797$ and $y_A = 183.0674$, respectively. Alternatively, if *P* were the local benchmark, the coordinates of *O* and *A* would be $x_O = –46.4030$ and $y_O = –21.6381$, and $x_A = –14.1232$ and $y_A = 1613.4293$. (Note that one cannot think of coordinates independently of the origin, which is the point of reference.)

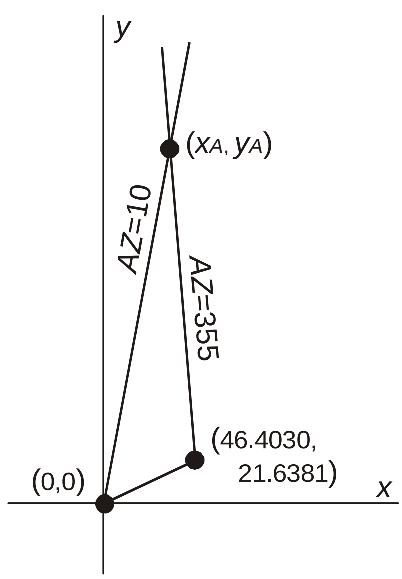

Figure 6. Graph showing information to determine position of *A* using linear equations.

Relative to a coordinate origin at *O*, the *x*- and *y*-coordinates of *A* are 21.6381 and 32.2797, as has just been stated. This is a perfectly acceptable answer to our original question (Where is *A*?) if units (m) are added to the numbers. The route we took to obtain this answer involved finding the length *OA* from the law of sines and, then, the components of the vector $\mathbf{v}_{\mathbf{O\text{-}A}}$ from its magnitude and direction. The coordinates x_A and y_A can also be found directly.

Each of the bearings is a *line* of sight, and we can find the *equation of those lines*. The location of Point *A*, which is at the intersection of these two lines, can be found by solving the two simultaneous equations for the point they have in common (x_A and y_A). Thus, we need to find (Fig. 6): the equation of the line that passes through (0, 0) and (x_A and y_A); the equation of the line that passes through (46.4030, 21.6381) and (x_A and y_A); and the solution of those two simultaneous equations.

The first part of the problem is an exercise in school algebra (Fig. 7): What is the equation of a line given the slope (*m*) and the location of one of the points (x_1, y_1)? We know the

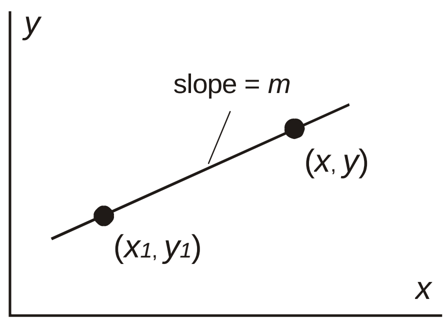

Figure 7. Graph showing the essentials of a two-point problem.

slope of the line because we know its bearing. The slope of the line given its mathematical direction (θ) is

$$m = \tan\theta\,. \tag{10a}$$

Working with azimuths (*AZ*), it is more convenient to use

$$m = \frac{1}{\tan(AZ)}\,, \tag{10b}$$

which follows from Equations A1 (Appendix) and 10a.

With *m*, the equation of the line can be found by rearranging the equation of the slope,

$$m = \frac{y - y_1}{x - x_1} \tag{11a}$$

to produce

$$y = mx + (y_1 - mx_1)\,, \qquad (11b)$$

from which, the intercept is

$$y_{intercept} = y_1 - mx_1\,. \qquad (12)$$

Turning to Figure 1, let $x_1 = x_O$ and $y_1 = y_O$; then with Equations 11a and 12, the line from *O* through *A* is:

$$y = x\tan\theta_{OA} + y_O - x_O\tan\theta_{OA}\,, \qquad (13a)$$

where θ_{OA} is the mathematical direction from O to A. Similarly, with $x_1 = x_P$ and $y_1 = y_P$, Equations 11a and 12 produce the line from *P* to *A* as

$$y = x\tan\theta_{PA} + y_P - x_P\tan\theta_{PA}\,, \qquad (13b)$$

where θ_{PA} is the mathematical direction from *P* to *A*. With $\theta_{OA} = 80°$, $x_O = 0$, $y_O = 0$, $\theta_{PA} = 95°$, $x_P = 46.4030$, and $y_P = 21.6381$ (Fig. 6), the equations of the two lines are

$$y = 5.6712x\,, \qquad (14a)$$

for *OA*, and

$$y = -11.4295x + 552.00\,, \qquad (14b)$$

for *PA*.

The problem now has been reduced to an exercise in solving two simultaneous equations in two unknowns, *x* and *y* (Fig. 8). This can be done easily by using Cramer's rule (see Computational Geology-12, Cramer's rule and the three-point problem, Sept. 2000). First, the equations need to be rearranged to Cramer's rule form:

$$\begin{aligned} \alpha_{11}x + \alpha_{12}y &= \beta_1 \\ \alpha_{21}x + \alpha_{22}y &= \beta_2 \end{aligned} \qquad (15)$$

Then, the answers can be written down immediately from

$$x = \frac{D_x}{D} \text{ and } y = \frac{D_y}{D}\,, \qquad (16)$$

where D, D_x, and D_y are the determinants

$$D = \begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{vmatrix}\,, \qquad (17a)$$

$$D_x = \begin{vmatrix} \beta_1 & \alpha_{12} \\ \beta_2 & \alpha_{22} \end{vmatrix}, \tag{17b}$$

and $$D_y = \begin{vmatrix} \alpha_{11} & \beta_1 \\ \alpha_{21} & \beta_2 \end{vmatrix}. \tag{17c}$$

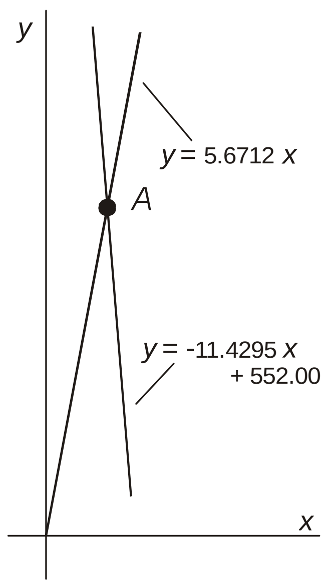

Figure 8. Graph showing the two lines that find that the position of *A*.

In this case, the simultaneous equations in Cramer's rule form are:

$$\begin{aligned} 5.6712x - y &= 0 \\ -11.4295x - y &= -552.00 \end{aligned}. \tag{18}$$

The solution determinants, then, are:

$$D = \begin{vmatrix} 5.6712 & -1 \\ -11.4295 & -1 \end{vmatrix} = -17.1007; \tag{19a}$$

$$D_x = \begin{vmatrix} 0 & -1 \\ -552.00 & -1 \end{vmatrix} = -552.00; \tag{19b}$$

$$D_y = \begin{vmatrix} 5.6712 & 0 \\ -11.4295 & -552.00 \end{vmatrix} = -3130.51. \tag{19c}$$

The solution is

$$x = \frac{-552.00}{-17.1007} = 32.279\,, \tag{20a}$$

and
$$y = \frac{-3130.51}{-17.1007} = 183.063\,. \tag{20b}$$

These values are x_A and y_A, respectively, the point *A* where the lines cross (compare with Table 1, $\mathbf{v_{O\text{-}A}}$).

A Mapping Problem

Determining the coordinates of the various points of interest in a mapped area opens the door for calculation of many other quantities. For example, consider this problem: Suppose *A* in Figure 2 is simply the first of four corners of a parcel of land. Suppose that just as you can see *A* from *O* and *P*, you can also see the three other corners of the property from *O* and *P*. Suppose you make the sightings on these other corners, and the data are as in Figure 9. :

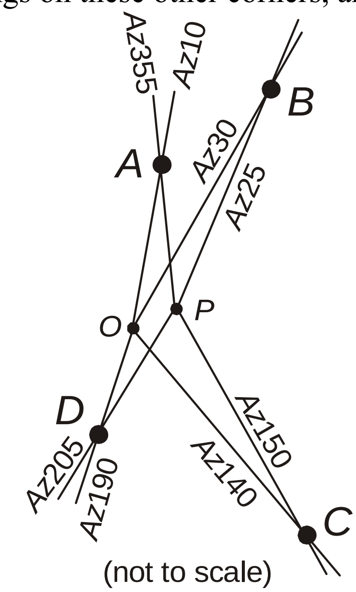

Figure 9. Sketch showing sightings to locate the four vertices of a quadrilateral from a single baseline (*OP*).

Then:

- What is the area of the parcel of land (*ABCD*)?
- What is the perimeter?
- What are the angles at the four distant corners?

Understanding the Problem and Devising a Plan.

Figure 9 shows the direction data. Figure 10 shows the location information: two points (*O* and *P*) whose location is known (again, with excess precision), and four points (*A*, *B*, *C*, and *D*) whose location must be learned from the sightings. If the coordinates of these four points were known, one could plot their positions on graph paper, connect the dots and answer the questions graphically:

- By counting the squares within the quadrilateral, *ABCD*.
- By measuring the length of all the sides.
- By measuring the angles with a protractor.

If the answers can be determined graphically, they can also be determined using vectors. Specifically, the answers to our questions can be found:

- By using the *cros-product* of vectors to determine areas (Computational Geology 14, The vector cross-product and the three-point problem**,** Jan. 2001).
- By finding *the magnitude* of vectors (Equation 6a) to determine lengths.
- By using the *dot product* of vectors to determine angles (CG-4).

We will take them one by one.

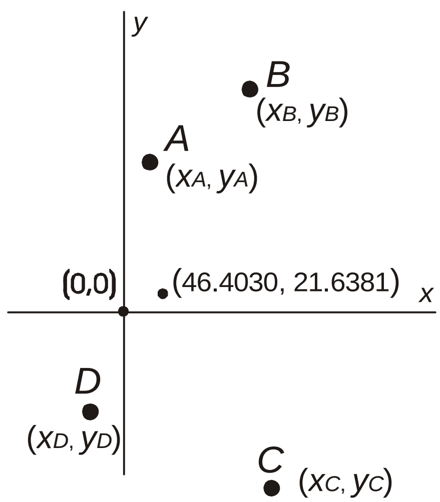

Figure 10. Sketch showing the coordinates of the end-points of the baseline and the vertices of the problem quadrilateral. Extra digits for *P* for purpose of calculation.

The cross-product and the area of *ABCD*. The magnitude of the vector cross-product **u** × **v** is the area of the parallelogram formed by **u** and **v** (Fig. 8 of CG-14**).** Therefore, the areas of triangle *DBC* (A_{DBC}) and triangle *DAB* (A_{DAB}) in Figure 11 are

$$A_{DBC} = |(\mathbf{v_{D\text{-}C}} \times \mathbf{v_{D\text{-}B}})| / 2 \tag{21a}$$

and

$$A_{DAB} = |(\mathbf{v_{D\text{-}B}} \times \mathbf{v_{D\text{-}A}})| / 2 , \tag{21b}$$

respectively, and the area of the quadrilateral (A_{ABCD}) is their sum

$$A_{ABCD} = A_{DBC} + A_{DAB} . \tag{22}$$

For vectors **u** and **v** in the *xy*-plane and arranged such that **u** rotates counterclockwise through an acute angle into **v** (i.e., such that the direction of **u** × **v** is parallel to the *z*-axis, according to the right hand rule – as is the case here), |**u** × **v**| is calculated easily from the determinant consisting of the vector components:

$$|\mathbf{u} \times \mathbf{v}| = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = u_x v_y - u_y v_x \,. \qquad (23)$$

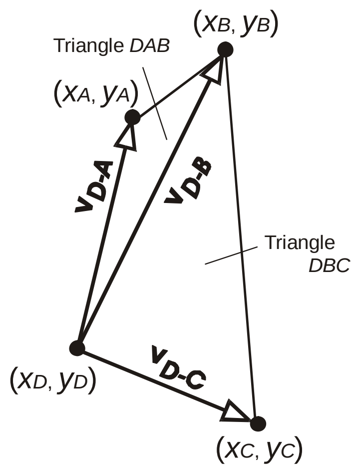

Figure 11. Sketch showing vectors to find the area of the quadrilateral using the cross-product (Equations 21 and 22).

The magnitude of vectors and the perimeter of *ABCD*. The perimeter of *ABCD* (P_{ABCD}) is the sum of the four vectors that go from one corner to the next around the quadrilateral (Fig. 12):

$$P_{ABCD} = |\,\mathbf{v_{D\text{-}A}}\,| + |\,\mathbf{v_{A\text{-}B}}\,| + |\,\mathbf{v_{B\text{-}C}}\,| + |\,\mathbf{v_{C\text{-}D}}\,| \,. \qquad (24)$$

The individual magnitudes are calculated easily from the Pythagorean sum of the vector components (Equation 6a).

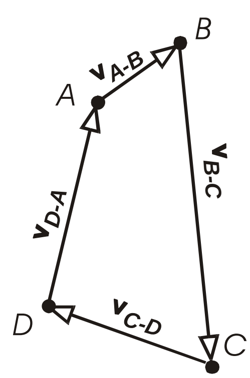

Figure 12. Sketch showing the vectors to find the perimeter of the quadrilateral using vector addition (Equation 24).

The dot product of vectors and the angles at *A, B, C* and *D*. The dot product **u·v**, is the scalar

$$\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\cos\theta, \qquad (25)$$

where θ is the angle formed by **u** and **v**. Thus, for example (Fig. 13),

$$\mathbf{v}_{\mathbf{D\text{-}C}} \cdot \mathbf{v}_{\mathbf{D\text{-}A}} = |\,\mathbf{v}_{\mathbf{D\text{-}C}}\,|\,|\,\mathbf{v}_{\mathbf{D\text{-}A}}\,|\cos\theta_{ADC}\,. \qquad (26)$$

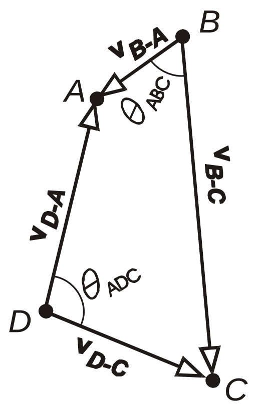

Figure 13. Sketch showing vectors to find the vertex angles at *B* and *D* using the dot product (Equation 26).

Rearranging Equation 26 produces an easy way to calculate θ_{ADC}. The dot product **u** · **v** is easily found from the components of **u** and **v** by (Equation 32 of CG-4):

$$\mathbf{u}\cdot\mathbf{v} = u_x v_x + u_y v_y\,. \qquad (27)$$

Getting and using the components. It is evident from Equations 6a, 23 and 27 that we need to be able to write down the components of all the vectors in order to carry out the vector calculations. The components are easily found from the coordinates of *A*, *B*, *C* and *D*. Thus, as

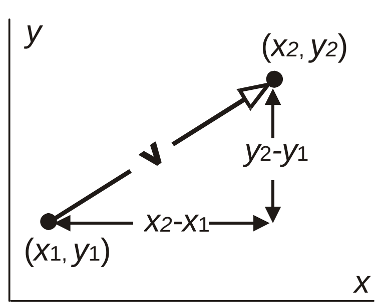

Figure 14. Diagram showing the terms in Equation 28a.

in the general case shown in Figure 14, the vector **v** from (x_1,y_1) to (x_2, y_2) is simply

$$\mathbf{v} = (\Delta x)\,\mathbf{i} + (\Delta y)\,\mathbf{j} = (x_2 - x_1)\,\mathbf{i} + (y_2 - y_1)\,\mathbf{j}\,, \tag{28a}$$

where Δx and Δy are the components v_x and v_y, respectively. For our problem involving vectors that go from one vertex to another around or across the quadrilateral, we can use a notation for the components as in the following example for the vector from A to B

$$\mathbf{v_{A\text{-}B}} = \Delta x_{\mathrm{AB}}\,\mathbf{i} + \Delta y_{\mathrm{AB}}\,\mathbf{j}\,. \tag{28b}$$

Then from Equations 21, 22, and 28b, the area A_{ABCD} is

$$A_{ABCD} = \frac{\begin{vmatrix} \Delta x_{DC} & \Delta y_{DC} \\ \Delta x_{DB} & \Delta y_{DB} \end{vmatrix}}{2} + \frac{\begin{vmatrix} \Delta x_{DB} & \Delta y_{DB} \\ \Delta x_{DA} & \Delta y_{DA} \end{vmatrix}}{2}\,. \tag{29}$$

From Equations 6a, 24 and 28b, the perimeter P_{ABCD} is

$$P_{ABCD} = \sqrt{\Delta x_{DA}^2 + \Delta y_{DA}^2} + \sqrt{\Delta x_{AB}^2 + \Delta y_{AB}^2} + \sqrt{\Delta x_{BC}^2 + \Delta y_{BC}^2} + \sqrt{\Delta x_{CD}^2 + \Delta y_{CD}^2}\,. \tag{30}$$

(Note, Δx_{DA}^2 means $(x_A - x_D)^2$ and not $x_A^2 - x_D^2$.) Finally, from Equations 6a, 26, 27 and 28b, the angle at the southwest corner θ_{ADC} is

$$\theta_{ADC} = \cos^{-1} \frac{\Delta x_{DC}\Delta x_{DA} + \Delta y_{DC}\Delta y_{DA}}{\sqrt{\Delta x_{DC}^2 + \Delta y_{DC}^2}\,\sqrt{\Delta x_{DA}^2 + \Delta y_{DA}^2}}\,. \tag{31}$$

A plan. The problem can be solved by carrying out the following three-step plan:

1. Find the coordinates of the four vertices by triangulation.
2. Find the coordinates of the vectors that go from vertex to vertex around the perimeter and across the diagonals using appropriate forms of Equations 28.
3. Do the vector arithmetic. Specifically:
   a. Find the area from the cross-products using Equation 29.
   b. Find the perimeter from the magnitude of vertex-to-vertex vectors using Equation 30.
   c. Find the vertex angles from the dot product using Equation 31 for the southwest vertex and similar equations for the other vertices.

Carrying out the Plan

Figure 15 shows a spreadsheet that makes the calculations for the first part of the problem: the four triangulation exercises of Figure 9.

	B	C	D	E	F	G	H	I	J
2	**A MAP PROBLEM USING VECTORS**								
3			**page 1: Triangulating the four corners**						
4	**DATA**								
5	**A. Coordinates of baseline points (m)**				*x*	*y*			
6				O	0	0			
7				P	46.4030	21.6381			
8	**B. Sightings to the four corners (Azimuths in degrees)**								
9					A	B	C	D	
10				From O	10	30	140	190	
11				From P	355	25	150	205	
12									
13	**FINDING COORDINATES OF THE CORNERS**								
14									
15	**A. Equations of lines from baseline points to corners**								
16			slope	intercept				slope	intercept
17		O to A	5.6712	0			O to C	-1.1918	0
18		P to A	-11.4295	552.00			P to C	-1.7320	102.01
19		O to B	1.7320	0			O to D	5.6713	0
20		P to B	2.1445	-77.87			P to D	2.1445	-77.87
21									
22	**B. Equations into Cramer's Rule form**								
23		*x*-coeff	*y*-coeff	right side			*x*-coeff	*y*-coeff	right side
24	O to A	5.6712	-1	0		O to C	-1.1918	-1	0
25	P to A	-11.4295	-1	-552.00		P to C	-1.7320	-1	-102.01
26	O to B	1.7320	-1	0		O to D	5.6713	-1	0
27	P to B	2.1445	-1	77.87		P to D	2.1445	-1	77.87
28									
29	**C. Determinants for Cramer's Rule**								
30	For A:	D	5.671243	-1		For C:	D	-1.19	-1
31			-11.42954	-1				-1.73	-1
32									
33		Dx	0	-1			Dx	0	-1
34			-552.003	-1				-102.01	-1
35									
36		Dy	5.671243	0			Dy	-1.19	0
37			-11.42954	-552.00				-1.73	-102.01
38									
39	For B:	D	1.73	-1		For D:	D	5.67	-1
40			2.14	-1				2.14	-1
41									
42		Dx	0	-1			Dx	0	-1
43			77.87	-1				77.87	-1
44									
45		Dy	1.73	0			Dy	5.67	0
46			2.14	77.87				2.14	77.87
47									
48	**D. Results**		*x*	*y*				*x*	*y*
49		A	32.279	183.064			C	188.805	-225.008
50		B	188.804	327.018			D	-22.081	-125.226

Figure 15. Spreadsheet for calculating the coordinates of the four vertices of the problem quadrilateral (four triangulation exercises).

The data are in Rows 6-11. The algorithm in the spreadsheet does the following:

- Calculates the slope and intercept of the eight lines of sight in Rows 17-20 using Equations 10b and 12, respectively. (The example in Equations 14 is in Columns D and E of Rows 17 and 18.)
- Converts the equations of the lines from the slope-and-intercept form of Rows 17-20 to the linear-coefficients form appropriate for Cramer's rule (Equations 15) in Rows 24-27. (The example in Equations 18 is in Columns C, D, and E of Rows 24 and 25.)
- Forms the Cramer's rule determinants (Equations 17) in Rows 30-46. (The determinants corresponding to Equations 19 are in the Block D30:E37.)
- Calculates the ratios of determinants that produce the coordinates of *A*, *B*, *C* and *D* in Rows 49 and 50 (Equations 20). (The result we obtained earlier for *A* is in Block D49:E49.)

Figure 16 shows a spreadsheet that works out the rest of the problem

	B	C	D	E	F	G	H	I	J
2	**A MAP PROBLEM USING VECTORS**								
3			**page 2: Finding the area, perimeter and vertex angles**						
4	**DATA**								
5	**Coordinates of corners**			*x*	*y*			*x*	*y*
6			A	32.279	183.064		C	188.805	-225.008
7			B	188.804	327.018		D	-22.081	-125.226
8									
9	**FINDING THE POINT-TO-POINT VECTORS**								
10	peripheral		*x*-cmpnt	*y*-cmpnt			*x*-cmpnt	*y*-cmpnt	
11		v A to B	156.53	143.95		v B to A	-156.53	-143.95	
12		v B to C	0.00	-552.03		v C to B	0.00	552.03	
13		v C to D	-210.89	99.78		v D to C	210.89	-99.78	
14		v D to A	54.36	308.29		v A to D	-54.36	-308.29	
15									
16	diagonals								
17		v A to C	156.53	-408.07		v C to A	-156.53	408.07	
18		v B to D	-210.88	-452.24		v D to B	210.88	452.24	
19									
20	**FINDING DISTANCES**					**FINDING DIRECTION**			
21		A to B	212.66	m		(Using ATAN2 function)			
22		B to C	552.03	m			theta	Azimuth	
23		C to D	233.30	m		A to B	42.60	47.40	
24		D to A	313.05	m		B to C	-90.00	180.00	
25						C to D	154.68	295.32	
26		A to C	437.06	m		D to A	80.00	10.00	
27		D to B	499.00	m					
28									
29		Perimeter	1311	m					
30									
31	**FINDING AREA**								
32									
33		Triangle DCB: v D to C cross v D to B							
34			Determinant					Areas	
35			210.89	-99.78					
36			210.88	452.24	=	116414		58207	m2
37		Triangle CAB: v D to B cross v D to A							
38			Determinant						
39			210.88	452.24					
40			54.36	308.29	=	40430		20215	m2
41									
42		Total area						78422	m2
43									
44									
45	**FINDING CORNER ANGLES**								
46				dot-prodct	mag v1	mag v2	cos angle	angle (deg)	
47	NW	v A to D dot v A to B		-52888.3	313.0	212.7	-0.7945	142.60	
48	NE	v B to A dot v B to C		79466.4	212.7	552.0	0.6769	47.40	
49	SE	v C to B dot v C to D		55082.5	552.0	233.3	0.4277	64.68	
50	SW	v D to C dot v D to A		-19298.2	233.3	313.0	-0.2642	105.32	

Figure 16. Spreadsheet for calculating the length and direction of edges, perimeter, area, and vertex angles of the quadrilateral from the coordinates of the vertices.

The coordinates of the four corners are repeated in Rows 6 and 7 from Figure 15. The algorithm does the following:

- Calculates the components of the vertex -to- vertex vectors in Rows 11-18 using Equations 28 and the coordinates in Rows 6 and 7.
- Calculates the magnitude of the vectors in Rows 21-27 using Equation 6a and the components in Rows 11-18.
- Finds the direction of the peripheral vectors from the components in Rows 11-14 by using the cell formulas of Columns F and H of the spreadsheet in the Appendix.
- Finds the perimeter in Cell D29 by summing the magnitudes in Block D21:D24.
- Calculates the area in Rows 33-42 by using Equation 29 and the components in Rows 11-18.

Calculates the vertex angles in Rows 47-50 using Equation 27 in Column E, Equation 6a in Columns F and G, and Equation 26 in Columns H and I, with the components in Rows 11-14.

The answers are shown in Cells I42, D29 and I47 to I50: the area is 78,422 m^2, the perimeter is 1311 m, and the four angles are 142.60°, 47.40°, 64.68°, and 105.32°. Note that the four angles sum to 360°, as they should.

Looking Back.

Some checks. There are various things one can do to check the reasonableness of the answers. The first thing is to draw the quadrilateral more carefully to scale using the coordinates and annotate the drawing with the calculated length and direction information (Fig. 17). Do the azimuths and relative magnitudes look right?

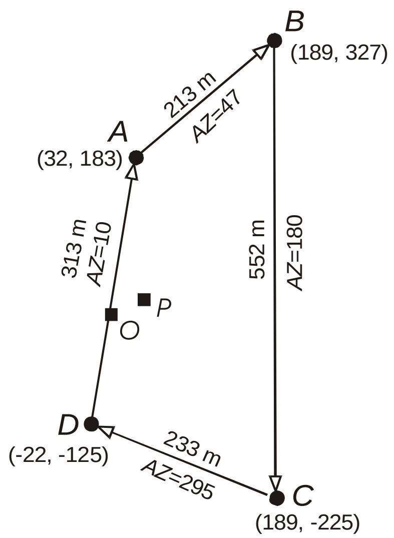

Figure 17. Diagram of the quadrilateral drawn to scale to check for calculation errors.

In case it was not apparent before, it is obvious from Figure 17 that the western edge of the quadrilateral passes through the origin (*O*). This result checks with the starting information: the azimuth from *O* to *A* is 10° and the azimuth from *O* to *D* is 190°. Perhaps we should have spotted that when sketching Figure 9.

It is also evident from Figure 17 (and the *x*-coordinates in Cells E7 and I6 of Fig. 16) that the eastern edge runs due north-south. That is so, if and only if

$$\frac{\tan\theta_{O-B}}{\tan\theta_{O-C}} = \frac{y_B}{y_C}, \tag{32}$$

where $\theta_{O\text{-}B}$ and $\theta_{O\text{-}C}$ refer to the directions of $\mathbf{v_{O\text{-}B}}$ and $\mathbf{v_{O\text{-}C}}$, respectively. Both ratios are −1.453.

Another test can be done easily by checking whether the calculated directions of the edges (Fig 16, Rows 23-26 of Column I) are consistent with the vertex angles (Fig. 16, Rows 47-

50 of Column I). The relevant information is in Figure 18. Consider the southeast vertex (*C*), for example. The angle between the two labeled azimuths is 65°, which is the same as the vertex angle; the two must be the same because they are on opposite sides of intersecting lines. The northeast and southwest vertices check out in the same way. For the northwest vertex, the angle between the labeled azimuths is 37°, the supplement of the vertex angle (143°).

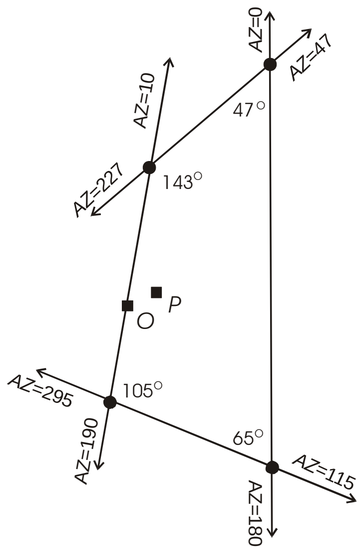

Figure 18. Diagram used to compare direction information with vertex angles.

As a last check, we can determine the area of the quadrilateral trigonometrically using the information shown in Figure 19: the coordinates of the corners (Fig. 15, Rows 49 and 50) and the lengths of the edges (Fig. 16, Col. D, Rows 21-24).

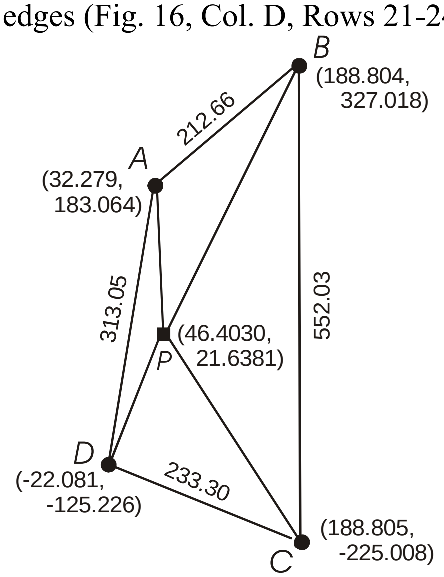

Figure 19. Diagram used to find the central angles about *P* and the area of the quadrilateral trigonometrically. Extra digits for purpose of comparison.

We can divide the quadrilateral into triangles radiating out from *P*. The radiating legs can be found from the coordinates using the Pythagorean sum (Equation 6a): *PA* = 162.0429 m, *PB* = 336.9499 m, *PC* = 284.8028 m, and *PD* = 162.046 m. Then the four central angles at *P* can be calculated using the *law of cosines*:

$$c^2 = a^2 + b^2 - 2ab\cos\gamma \,, \qquad (33)$$

where γ is the included angle between sides *a* and *b*, and *c* is the side opposite. (Let γ = 90°, and Equation 33 reduces to the familiar Pythagorean theorem). Thus, solving for γ and plugging in the lengths for *PA*, *PB*, *PC*, and *PD*, the four angles are (clockwise, starting from the northern triangle): ∠*APB* = 30.000°, ∠*BPC* = 125.000°, ∠*CPD* = 55.000°, and ∠*DPA* = 150.000°). These angles check with the differences of the azimuths for the original sightings from *P* (Fig. 15, Row 11).

Finally, the areas of the four triangles can be calculated from

$$Area = \frac{ab\sin\gamma}{2} \,. \qquad (34)$$

Clockwise from the northern triangle, the areas work out as follows: *APB* = 13,650 m^2, *BPC* = 39,305 m^2, *CPD* = 18,902 m^2, and *DPA* = 6565 m^2. These four areas sum to 78,422 m^2 (compare with Figure 16, Cell I 42).

Significant digits. As a last consideration, we can think about the number of digits in the calculated area. Cell I42 shows the calculation to 1 m – five significant digits. Surely, that is more digits than are warranted. Now that we have spreadsheets for the calculations, we can easily investigate how many would be appropriate.

Suppose that each of the original sightings can be incorrect (inaccurate) by as much as 1°. Let's say that they are incorrect in such a way that the corners are moved further away from *O* and *P* (thus creating a larger quadrilateral). Any vertex can be pushed away by decreasing the triangulation angle at the far corner (*OAP* in Figure 2). For the vertices at *A* and *B* (Fig 9), this is achieved with a counterclockwise rotation of the sighting from *O* and a clockwise rotation of the sighting from *P*; for the vertices at *C* and *D,* it is achieved with a clockwise rotation of the sighting from *O* and a counterclockwise rotation of the sighting from *P*. Thus all the corners can be moved out by replacing the azimuths in Row 10 of Figure 15 with 9, 29, 141 and 191, and the azimuths in Row 11 of Figure 15 with 356, 26, 149, 204.

The result of these changes is as follows. The new coordinates (replacing Rows 49 and 50 of Fig. 15) are: *A*, (33.240, 209.867); *B*, (298.5, 538.5); C, (230.3, –284,3); and *D*, (–28.5, –146.6). (Why does *B* move so far?) The new area (in Cell I42 of Figure 16) is 148,276 m^2, an increase of 85%. Moving the corners closer in with 1° changes produces an area of 50,054 m^2, a 37% decrease. Therefore, given a ±1° uncertainty in the original sightings, the area might be stated as $80{,}000^{+70{,}000}_{-30{,}000}$ m^2.

Clearly, more than one digit misrepresents how well the calculated area is known. Even one significant digit implies less uncertainty than is actually present, because 80,000 implies 80,000 ± 5,000 (Taylor, 1997 and Computation Geology 1, Significant figures!, May 1998).

Concluding Remarks

Triangulation is a useful tool for finding locations. It is also fertile ground to practice school algebra and geometry and to explore the consequences of measurement errors. Uncertainty in calculated positions can be quite large for distant points (small triangulation angles), especially for sightings that make a low angle with the baseline. The example discussed here is artificial in that most people would change the position of the baseline for the sightings to *B* and *D* so that they could face those points at a better angle. There are all kinds of relationships that can be explored with the spreadsheets of Figures 15 and 16.

Reference

Taylor, J.R., 1997, An introduction to error analysis: the study of uncertainties in physical measurements, 2nd edition: Sausalito CA, University Science Books, 327 p.

Acknowledgment

In a discussion of spreadsheet vs. graphical approaches to geometric structural- geology problems at an NGAT workshop, Vince Cronin of the University of Wisconsin-Milwaukee pointed out that a large benefit of the spreadsheet approach is that one can easily examine the effects of uncertainties. This column illustrates Vince's point for a mapping problem.

Appendix

Bearings, azimuths and the ATAN2(*x*,*y*) function

In geology, there are two conventions for stating direction: bearing and azimuth. Bearings are acute angles measured clockwise or counterclockwise from N or S. For example, the bearing N20E is the direction 20° clockwise from N, and N20W is the direction 20° counterclockwise from N. Similarly, S20E indicates a direction 20° counterclockwise from S, and S20W indicates a direction 20° clockwise from S. If you omit the number of degrees, you can see immediately the general quadrant of the direction: NE, NW, SE, and SW in these examples.

Azimuths are angles ranging from zero to 360° and are measured clockwise from N. Thus, N20E, S20E, S20W, and N20W correspond to azimuths of 20°, 160°, 200°, and 340°, respectively. In practice, azimuths are positive; negative azimuths are frowned upon, although an azimuth of -20° can logically be considered the same as an azimuth of 340°.

In mathematics, the convention is to state the direction of a vector in terms of θ measured clockwise from the *x*-axis. Negative θ's are definitely allowed and refer to angles measured clockwise (i.e., negatively) from the *x*-axis. For map work, it is standard to let the *x*-axis be E, and the *y*-axis N, so that the map presents a Cartesian coordinate system with north at the top of the page. With that convention, the NE and NW quadrants are indicated by $\theta>0$, and the SW and SE quadrants are indicated by $\theta<0$.

A useful relation between θ and the azimuth (*AZ*) is

$$\tan\theta = \frac{1}{\tan(AZ)}\,. \qquad \text{(A1)}$$

In using vectors for directed distances on a map, one commonly needs to find the direction (θ or *AZ*) of **v** from its components, v_x and v_y. This inevitably means using the inverse tangent (Equation 6b), a.k.a the arctangent function. There is a problem, however: the tangent function repeats itself after 180°, only half of the full circle of map directions. For example, a vector pointing NE (e.g., $v_x = 1$ and $v_y = 1$) has the same tangent (1.000) as a vector pointing SW (e.g., $v_x = -1$ and $v_y = -1$). This means that the inverse tangent does not discriminate between opposite quadrants. If v_x/v_y is positive and one pushes the arctan key on a calculator, the answer will appear as a θ indicating the NE quadrant (θ between 0 and 90°), whether the vector really points NE or SW. Similarly, if v_x/v_y is negative, the calculator will produce a θ indicating the SE quadrant (θ between 0 and -90°), whether **v** points SE or NW.

One can easily sort out the two possibilities by sketching a little figure showing $v_x\mathbf{i}$, $v_y\mathbf{j}$, and their sum, **v**, and I strongly recommend that students draw such a figure. Alternatively, one can look at the sign of v_x. If v_x is negative then **v** is SW if v_x/v_y is positive, and NW if v_x/v_y is negative. If v_x is positive, then **v** is NE if v_x/v_y is positive, and SE if v_x/v_y is negative.

On a spreadsheet one can avoid the ambiguity of the inverse tangent by using the ACTAN2(x,y) function (see spreadsheet example below). The ATAN2 function has two arguments, x and y (which, in the case of our vector problems, are v_x and v_y). The function is programmed to look not only at the ratio of x/y but also at the sign of x.

Columns E and F of the spreadsheet example illustrate the advantages to our work of the ATAN2 function. Not only does the ATAN2 avoid the ambiguity of the one-argument ATAN function, but it also avoids the nuisance of dividing by zero when $v_x = 0$.

Because of Equation A1, one can reverse the arguments in the ATAN2 function and produce an angle very much like the azimuth (Column G). The only "problem" is that it produces negative azimuths for the two western quadrants (which really isn't much of a problem except by convention). To avoid negative azimuths, one can use the logic function illustrated in the footnote. This logic function says to add 360° to the result of the ATAN2(v_y, v_x) for vectors in the two western quadrants, otherwise take the result as is. The 180/3.14159 converts the result of the ATAN2(v_y, v_x) from radians to degrees.

	B	C	D	E	F	G	H
1	direction	coordinates			θ	"azimuth"	azimuth
2		v_x	v_y	ATAN(v_x/v_y)	ATAN2(v_x,v_y)	ATAN2(v_y,v_x)	* footnote
3	N	0	1	0	90	0	0
4	NE	1	1	45	45	45	45
5	E	1	0	#DIV/0!	0	90	90
6	SE	1	-1	-45	-45	135	135
7	S	0	-1	0	-90	180	180
8	SW	-1	-1	45	-135	-135	225
9	W	-1	0	#DIV/0!	180	-90	270
10	NW	-1	1	-45	135	-45	315
11	N	0	1	0	90	0	0

* Cell Formula for H3:
IF(C3<0, 360+ATAN2(D3,C3)*180/3.14159, ATAN2(D3,C3)*180/3.14159)