

Computational Geology 14

The Vector Cross-Product and the Three-Point Problem

H.L. Vacher, Department of Geology, University of South Florida, 4202 E. Fowler Ave., Tampa FL 33620

Topics this issue --

- *Mathematics: algebra – equation manipulation, linear interpolation, determinants, vectors; geometry – areas of triangles.*
- *Geology: three-point problem; geostatistics – point estimation; geochemistry – mixing lines and mixing triangles.*

Introduction

In CG-12 ("Cramer's Rule and the Three-Point Problem," Sept., 2000), we looked at the classic three-point problem of finding the strike and dip of a plane given the location and elevation of three points (Fig. 1). We used Cramer's Rule to find the equation of the plane through the three points; the equation of a horizontal line in the plane to find the direction of strike; and the gradient of the plane to determine the dip. Now we will consider a related problem: Given the elevation at three points on a plane, what is the elevation at any other point on the plane? The question is one of interpolation. As we will see, it is the three-dimensional upgrade of linear interpolation, which is familiar from school algebra. It also provides a tool for point estimation in geostatistics and mixing problems in geochemistry.

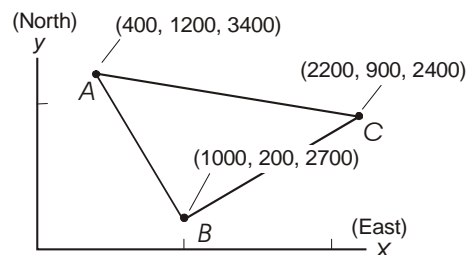


Figure 1. What is the strike and dip of the plane passing through these three points?

The Problem

A sample problem is shown in Figure 2. What is the elevation at the point marked by the asterisk, where $x = 1200$ ft and $y = 800$ ft, given the xyz -coordinates of A , B , and C ? The answer is easy because we know, from CG-12, how to find the equation of the plane through A , B , C . For this particular case, the equation of the plane is (CG-12, Equation 69)

$$z = 3110 - 0.488x + 0.407y . \quad (1)$$

We need only to substitute the values for x and y into Equation 1 to calculate z . The result is $z = 2850$ ft.

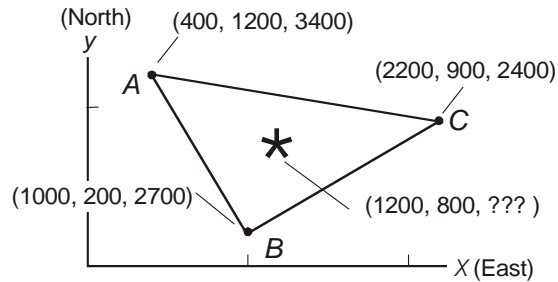


Figure 2. What is the elevation of the point indicated by the asterisk on the plane passing through the three corners?

Now let's consider the calculated result in more detail. The asterisk is clearly within the triangle, and the elevation of the point marked by the asterisk is clearly within the bounds of the other three elevations. Is the elevation an average of some sort?

In fact, the result, $z = 2850$ ft, is close to a simple average of the other three elevations. The average of 3400, 2700, and 2400 is 2833. Is it just a coincidence that the elevation is so close to the average? Perhaps not, because the asterisk is about at the exact center of the triangle. The center of the triangle (the centroid) is at $x = 1200$ ft, $y = 766.7$ ft. Plugging these central values into Equation 1 produces $z = 2830$ ft to the three significant digits of Equation 1. So, we can say that the elevation at the center of the triangle is the average of the elevation at the three corners, and if one is close to the center of the triangle, the elevation is close to the average of the three corner elevations. Is there a way of quantifying this? I mean, is there a way of determining the elevation at a point within the triangle by using an average related to the location of the point?

Understanding the Problem.

Consider Figure 3. Point 1 is at the center of the triangle. As we discussed, the elevation at Point 1 is the average of the elevations of A , B , and C .

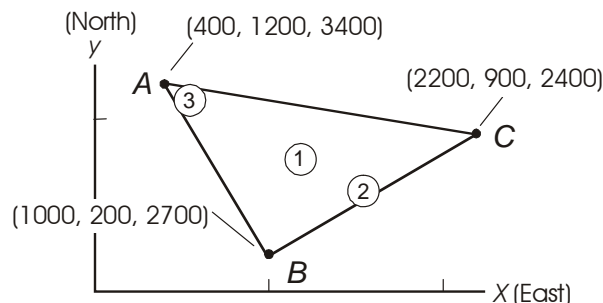


Figure 3. How does the elevation of the point located by the circled numbers vary from place to place as a function of the elevation of the three corners?

Point 2 is nearly on the line BC and about halfway between B and C . It is reasonable to expect that the elevation at Point 2 would be close to the average of the elevations at B and C . We can easily check this conjecture. The point midway between B and C is at $x = 1600$ and $y =$

550 ft. Using these values in Equation 1 produces $z = 2553$ ft, which is equal to the average of the elevations of B and C (2550 ft) to within the three significant digits.

Point 3 is nearly at A . It is reasonable to think its elevation will be very nearly that of A .

From these considerations of Figure 3, it is clear that we are looking for some sort of weighted average, where the elevations in the average are weighted according to their xy -location. If the point is close to one of the corners (Point 3), then the elevation at that corner should be weighted to dominate in the average. If the point is equidistant from two corners on the line between them (Point 2), the elevation of the corners should be weighted equally, and the elevation of the third corner should be weighted with a zero. If the point is in the center of the triangle, the elevations should all be weighted equally.

The equation for the weighted mean of the three elevations is:

$$z_{ave} = \frac{w_A z_A + w_B z_B + w_C z_C}{w_A + w_B + w_C}, \quad (2)$$

where w_i refers to the weights, z_i gives the elevations, and the subscripts refer to the corners. Because the total weight (w_T) is the sum Σw_i , Equation (2) can be restated as

$$z_{ave} = \omega_A z_A + \omega_B z_B + \omega_C z_C, \quad (3)$$

where ω_i refers to relative (or normalized) weights,

$$\omega_i = w_i / w_T. \quad (4)$$

We are looking for w_A , w_B , and w_C , and ω_A , ω_B , and ω_C . We need to devise a plan to find how these weights vary with the location of the interior point.

Devising a Plan, 1: The Analogous Problem in Two Dimensions

As we discussed in CG-12, the plane is the three-dimensional upgrade of a straight line in two dimensions, and, when trying to figure out a plane, a lot of good can come from looking at an analogous problem for a line.

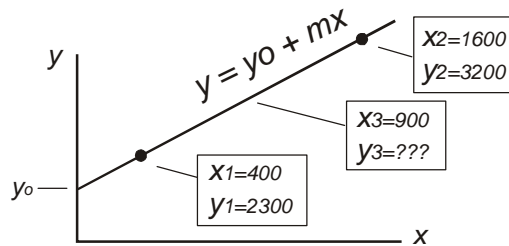


Figure 4. What is the value of y_3 on the straight line given the xy -coordinates of two points on the line and the value of x_3 ?

An example of the two-dimensional problem of finding a point on a line between two end-points is shown in Figure 4. The two end-points in this case are (400, 2300) and (1600, 3200) for

(x_1, y_1) and (x_2, y_2) , respectively. The question seeks the value of y_3 , given $x_3 = 900$. This question clearly asks for a linear interpolation.

Linear interpolation is easily done with

$$y_3 = y_1 + m(x_3 - x_1), \quad (5)$$

where m is the slope of the line connecting the end-points,

$$m = \frac{y_2 - y_1}{x_2 - x_1}. \quad (6)$$

Thus the equation for linear interpolation can be written as

$$y_3 = y_1 + \left(\frac{y_2 - y_1}{x_2 - x_1} \right) (x_3 - x_1). \quad (7)$$

Equation 7 can be rearranged (Appendix) to produce

$$y_3 = \frac{(x_2 - x_3)y_1 + (x_3 - x_1)y_2}{x_2 - x_1}. \quad (8)$$

Thus y_3 is the weighted average of y_1 and y_2 , where the weights are

$$w_1 = x_2 - x_3 \quad \text{and} \quad w_2 = x_3 - x_1, \quad (9)$$

and the sum of the weights is

$$w_T = x_2 - x_1. \quad (10)$$

The relative weights, therefore, are

$$\omega_1 = \frac{x_2 - x_3}{x_2 - x_1}, \quad \text{and} \quad \omega_2 = \frac{x_3 - x_1}{x_2 - x_1} = 1 - \omega_1. \quad (11)$$

The weighted average is

$$y_3 = \omega_1 y_1 + \omega_2 y_2. \quad (12)$$

Thus linear interpolation is the same as finding the weighted average of the two end-points where the weights correspond to Δx -distances between the various points. As shown by Equations 11, ω_1 approaches 1 and ω_2 approaches zero as the intermediate point (x_3, y_3) approaches the first end-point (x_1, y_1) . Similarly, as the point (x_3, y_3) approaches the other end-point (x_2, y_2) , ω_1 approaches zero, and ω_2 approaches 1.

A third way of solving the problem is to find the equation of the line in slope-and-intercept form (the two-point problem of CG-12, Equation 8), plug in the value of x_3 , and calculate y_3 :

$$y_3 = \left[y_1 - \frac{y_2 - y_1}{x_2 - x_1} x_1 \right] + \left(\frac{y_2 - y_1}{x_2 - x_1} \right) x_3. \quad (13)$$

Equation 13 is simply another rearrangement of Equations 7 and 8.

Regardless of which equation one chooses (Equations 7, 8, or 13), the answer to the question posed in Figure 4 is $y_3 = 2675$.

Devising a Plan, 2: Upgrading to a Plane

Geometrically, the relative weights ω_1 and ω_2 in Equation 11 are given by ratios of lengths of line segments shown in Figure 5. In Figure 5A, the line segments are parallel to the x -axis and correspond directly to the quantities of Equations 9-11. In Figure 5B, the line segments are along the line between the end-points, but clearly (by similar triangles), the ratios of these distances *along the line* are equal to the ratios of the corresponding distances along the x -axis. Therefore,

$$\omega_1 = \frac{s_2 - s_3}{s_2 - s_1}, \quad \text{and} \quad \omega_2 = \frac{s_3 - s_1}{s_2 - s_1} = 1 - \omega_1. \quad (14)$$

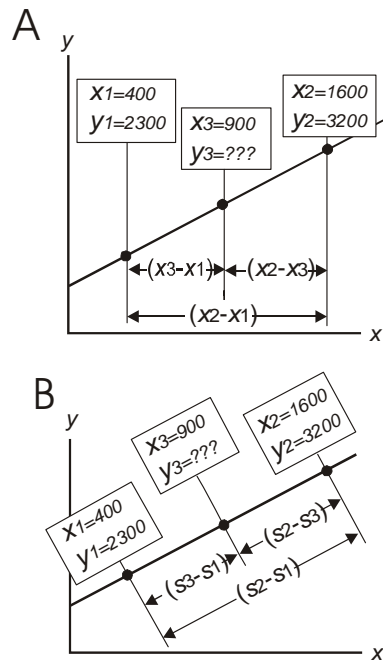


Figure 5. Distances of the intermediate point from the end-points (A) in the direction parallel to the x -axis and (B) along the line.

How does this upgrade to a plane?

We can start by restating Equations 14 to make the geometry of the question even more explicit. Call the two end-points P_1 and P_2 and the intermediate point P (Fig. 6). Then, Equations 14 become

$$\omega_1 = \frac{\overline{PP_2}}{\overline{P_1P_2}} \quad \text{and} \quad \omega_2 = \frac{\overline{P_1P}}{\overline{P_1P_2}}, \quad (15)$$

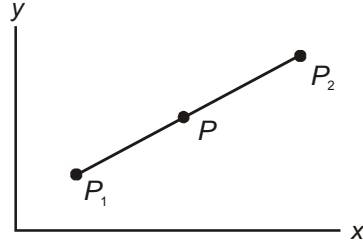


Figure 6. The general point P on a straight line connecting end-points P_1 and P_2 .

respectively, where $\overline{P_1P}$, $\overline{PP_2}$, and $\overline{P_1P_2}$ are lengths of the line segments, and the weighted average corresponding to Equation 8 is

$$y = \frac{\overline{PP_2}}{\overline{P_1P_2}} y_1 + \frac{\overline{P_1P}}{\overline{P_1P_2}} y_2. \quad (16)$$

Thus the value of y at the intermediate point is the average of the y -values at the end-points, weighted according to the length of the opposite line segment relative to the total length. (The same thing can be said in geology lingo. Geologists like the words "proximal" for "close" and "distant" for "far". So, assuming a vantage point close to P_1 in Figure 6, one can say that the value of y at the intermediate point is the y -value at the proximal point times the length of the distal line segment relative to the total distance plus the y -value at the distal point times the length of the proximal line segment relative to the total distance.) How does this geometric version of interpolation upgrade to a plane?

A geometric approach to the question is shown in Figure 7. In analogy to Figure 6 and Equation 16, we can conjecture that, given the xy -locations of A , B , and C ; the z -values of P_1 , P_2 , and P_3 ; and the xy -location of P within the triangle, then the z -value at P is the weighted average

$$z = \frac{A_{P_2PP_3}z_1 + A_{P_3PP_1}z_2 + A_{P_1PP_2}z_3}{A_{P_1P_2P_3}}, \quad (17)$$

where $A_{P_2PP_3}$, $A_{P_3PP_1}$, $A_{P_1PP_2}$, and $A_{P_1P_2P_3}$ refer to the areas of the triangles indicated in the subscripts. In other words, according to this conjecture, the relative weight for any given elevation is the fraction of the whole triangle that is covered by the area of the distal constituent triangle.

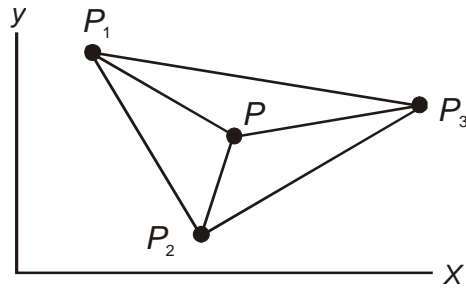


Figure 7. The general point P on a plane passing through P_1 , P_2 and P_3 . The triangle P_2P_3 is opposite the corner P_1 .

To prove this conjecture, we need to be able to calculate the area of triangles. The vector cross-product is well suited for this purpose.

Devising a Plan, 3: About the Cross-Product

In Computational Geology 4 ("Mapping with Vectors," Jan, 1999), we went over the basics of vectors: unit vectors; the calculation of magnitude and direction of a vector from its components; addition and subtraction of vectors; and one of the ways of multiplying two vectors. The multiplication that we discussed in CG-4 is the dot-product, $\mathbf{u} \cdot \mathbf{v}$, where \mathbf{u} and \mathbf{v} are vectors, and they are indicated as such by either bolding the font or adding an arrow overhead. The dot product results in a scalar and is commonly called the *scalar product*. It is useful for, among other things, finding angles.

The other type of vector multiplication is the *cross-product*, denoted by $\mathbf{u} \times \mathbf{v}$. The cross-product results in a vector and is commonly called the *vector product*. It is useful for, among other things, finding areas of triangles.

The cross-product $\mathbf{u} \times \mathbf{v}$ is defined as the *vector* with (a) *magnitude* equal to the *area* of the parallelogram that has \mathbf{u} and \mathbf{v} as sides (Fig. 8), and (b) *direction* given by the famous *right-hand rule*. The right-hand rule says that $\mathbf{u} \times \mathbf{v}$ is (a) perpendicular to the plane containing \mathbf{u} and \mathbf{v} and (b) oriented in the same direction as your thumb when you hold your *right* hand in a vertical plane above \mathbf{u} with fingers extended in the direction of \mathbf{u} , and then curl your fingers toward \mathbf{v} . (Note that if you curl the fingers of your right hand from \mathbf{v} to \mathbf{u} , your thumb points the other way; hence $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$, meaning they have the same magnitude but opposite direction.)

From the definition of the vector product, the magnitude of $\mathbf{u} \times \mathbf{v}$ is:

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}| |\mathbf{v}| \sin \theta_{uv}, \quad (18)$$

where $|\mathbf{u}|$ and $|\mathbf{v}|$ are the magnitudes of \mathbf{u} and \mathbf{v} , respectively, and θ_{uv} is the angle between \mathbf{u} and \mathbf{v} . That means that the area of the triangle formed from the vectors \mathbf{u} and \mathbf{v} as shown in the lower panel of Figure 8 is 1/2 of the magnitude of $\mathbf{u} \times \mathbf{v}$:

$$A_{triangle} = |\mathbf{u} \times \mathbf{v}| / 2, \quad (19)$$

where the vertical bars denote the magnitude of the vector cross-product.

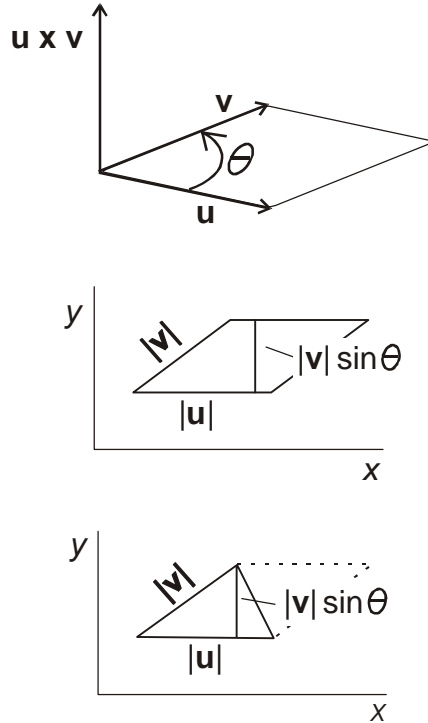


Figure 8. The vector cross-product $\mathbf{u} \times \mathbf{v}$ has magnitude equal to the area of a parallelogram with sides $|\mathbf{u}|$ and $|\mathbf{v}|$ and is perpendicular to the plane containing \mathbf{u} and \mathbf{v} . Half the magnitude of $\mathbf{u} \times \mathbf{v}$ is the area of a triangle with sides $|\mathbf{u}|$ and $|\mathbf{v}|$.

The magnitude of $\mathbf{u} \times \mathbf{v}$ typically is not calculated from Equation 18 but rather from a determinant consisting of unit vectors and the components of \mathbf{u} and \mathbf{v} . Determinants were reviewed in CG-12 in connection with Cramer's Rule and the solution of simultaneous linear equations. Vector products are another place where determinants are very useful.

The determinant expression for a vector product is as follows. The vectors \mathbf{u} and \mathbf{v} are individually represented as the vector sum of their component vectors:

$$\mathbf{u} = u_x \mathbf{i} + u_y \mathbf{j} + u_z \mathbf{k} \quad \text{and} \quad \mathbf{v} = v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k}. \quad (20)$$

Then

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix}. \quad (21)$$

Expanding this 3×3 determinant across its top row (CG-12, Equation 35) produces

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} u_y & u_z \\ v_y & v_z \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_x & u_z \\ v_x & v_z \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \mathbf{k}, \quad (22)$$

which makes evident the vector components of the cross-product. Even more explicitly, the 2×2 determinants can be expanded to give:

$$\mathbf{u} \times \mathbf{v} = (u_y v_z - u_z v_y) \mathbf{i} - (u_x v_z - u_z v_x) \mathbf{j} + (u_x v_y - u_y v_x) \mathbf{k}. \quad (23a)$$

from which, $\mathbf{u} \times \mathbf{v} = (u_y v_z - u_z v_y) \mathbf{i} + (u_z v_x - u_x v_z) \mathbf{j} + (u_x v_y - u_y v_x) \mathbf{k}.$ (23b)

(note the changes in the second term).

For the special case where both \mathbf{u} and \mathbf{v} are in the xy -plane, u_z and v_z are zero. In that case, Equations 22 and 23 become

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \mathbf{k} = (u_x v_y - u_y v_x) \mathbf{k}. \quad (24)$$

From Equation 24, the magnitude of the cross-product of two vectors in the xy -plane, then, is

$$|\mathbf{u} \times \mathbf{v}| = u_x v_y - u_y v_x. \quad (25)$$

From Equations 19 and 25, the area of the triangle formed on two sides by vectors \mathbf{u} and \mathbf{v} in the xy -plane (Fig. 8, lower panel) is

$$A_{triangle} = \frac{\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}}{2} = \frac{u_x v_y - u_y v_x}{2}. \quad (26)$$

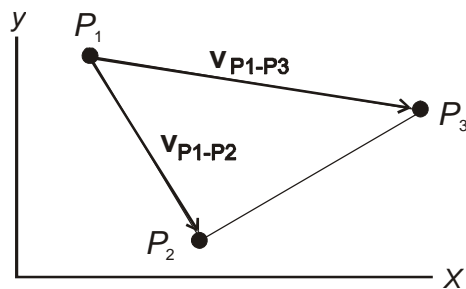


Figure 9. Vectors from P_1 to P_2 and from P_1 to P_3 .

Equation 26 can easily be recast in terms of the coordinates of the corners P_1 , P_2 , and P_3 , located at (x_1, y_1) , (x_2, y_2) , and (x_3, y_3) , respectively (Fig. 9). Thus, from Equation 20, the two-dimensional vectors \mathbf{v}_{P1-P2} and \mathbf{v}_{P1-P3} are:

$$\mathbf{v}_{P1-P2} = (x_2 - x_1) \mathbf{i} + (y_2 - y_1) \mathbf{j} \quad \text{and} \quad \mathbf{v}_{P1-P3} = (x_3 - x_1) \mathbf{i} + (y_3 - y_1) \mathbf{j}, \quad (27)$$

where $\mathbf{v}_{P_1-P_2}$ and $\mathbf{v}_{P_1-P_3}$ are read "the vector from P_1 to P_2 " and "the vector from P_1 to P_3 ," respectively. Then

$$A_{P_2P_1P_3} = |(\mathbf{v}_{P_1-P_2} \times \mathbf{v}_{P_1-P_3})| / 2 \quad (28)$$

is given in components as

$$A_{P_2P_1P_3} = \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix}}{2}. \quad (29)$$

Devising a Plan, 4: Finding the Weights

With the relationship between the area of triangles and the magnitude of cross-products, the areas of the internal triangles in the conjecture of Equation 17 are (Fig. 10A):

$$A_{P_2PP_3} = |(\mathbf{v}_{P-P_2} \times \mathbf{v}_{P-P_3})| / 2,$$

$$A_{P_3PP_1} = |(\mathbf{v}_{P-P_3} \times \mathbf{v}_{P-P_1})| / 2, \text{ and}$$

$$A_{P_1PP_2} = |(\mathbf{v}_{P-P_1} \times \mathbf{v}_{P-P_2})| / 2. \quad (30)$$

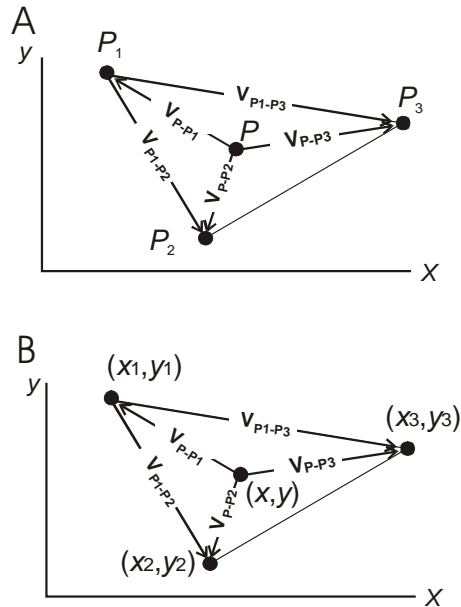


Figure 10. Vectors along the sides of triangles connecting P , P_1 , P_2 and P_3 .

With the relationship between cross-products and determinants, the areas of the internal triangles are (Fig 10B):

$$\begin{aligned}
A_{P_2PP_3} &= \frac{1}{2} \begin{vmatrix} x_2 - x & y_2 - y \\ x_3 - x & y_3 - y \end{vmatrix}, \\
A_{P_3PP_1} &= \frac{1}{2} \begin{vmatrix} x_3 - x & y_3 - y \\ x_1 - x & y_1 - y \end{vmatrix}, \text{ and} \\
A_{P_1PP_2} &= \frac{1}{2} \begin{vmatrix} x_1 - x & y_1 - y \\ x_2 - x & y_2 - y \end{vmatrix}.
\end{aligned} \tag{31}$$

Equations 17 and 31 provide a means of calculating z at a point within $P_1P_2P_3$, if our conjecture is correct that it is a weighted average of the z -values at the corners.

In terms of relative weights, the conjecture of Equation 17 is

$$z = \omega_1 z_1 + \omega_2 z_2 + \omega_3 z_3, \tag{32}$$

where the relative weights are (Fig. 7):

$$\begin{aligned}
\omega_1 &= \frac{A_{P_2PP_3}}{A_{P_1P_2P_3}}, \\
\omega_2 &= \frac{A_{P_3PP_1}}{A_{P_1P_2P_3}}, \text{ and} \\
\omega_3 &= \frac{A_{P_1PP_2}}{A_{P_1P_2P_3}}.
\end{aligned} \tag{33}$$

From Equations 31 and 33, and expanding the determinants in Equations 33, the relative weights become

$$\begin{aligned}
\omega_1 &= \frac{(x_2 - x)(y_3 - y) - (x_3 - x)(y_2 - y)}{(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)}, \\
\omega_2 &= \frac{(x_3 - x)(y_1 - y) - (x_1 - x)(y_3 - y)}{(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)}, \text{ and} \\
\omega_3 &= \frac{(x_1 - x)(y_2 - y) - (x_2 - x)(y_1 - y)}{(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)}.
\end{aligned} \tag{34}$$

In the end, our conjecture comes down to Equations 34. Are these correct expressions for the relative weights? In other words, do Equations 34 together with Equation 32 produce the value of z within the triangle?

As we said at the outset, there is another (known) way of producing the value of z within the triangle, namely plugging the coordinates of P into the equation of the plane found as part of the solution of the three-point problem in CG-12. Does this other way produce the same result?

The plan, now, is to manipulate appropriate equations of CG-12 into the form of a weighted average and see if the result is equivalent to Equations 32 and 34. If it is, then we will have shown that the z -value in the interior of the triangle is the average of the values at the corners, weighted according to the areas of the distal triangles.

Carrying out the Plan

Carrying out this plan provides a good exercise in expanding determinants. Given three sets of xy -coordinates – (x_1, y_1) , (x_2, y_2) , and (x_3, y_3) – the equation of a plane through them is (CG-12, Equation 57):

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0, \quad (35)$$

an equation that is easy to remember, and certainly well worth it.

Expanding the determinant in Equation 35 down the z -column, we get

$$z \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} - z_1 \begin{vmatrix} x & y & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} + z_2 \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} - z_3 \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0. \quad (36)$$

Solving for z ,

$$z = \frac{\begin{vmatrix} x & y & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} z_1 - \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} z_2 + \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} z_3}{\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}}. \quad (37)$$

Equation 37 is of the form of Equation 32:

$$z = \frac{D_1}{D} z_1 + \frac{D_2}{D} z_2 + \frac{D_3}{D} z_3, \quad (38)$$

where D_1 , D_2 , and D_3 are the determinants in the numerator of Equation 37, and D is the determinant in the denominator. Expanding the determinants of Equation 37, the ratios in Equation 38 are:

$$\begin{aligned} \frac{D_1}{D} &= \frac{xy_2 + x_2y_3 + x_3y - x_3y_2 - x_2y - xy_3}{x_1y_2 + x_2y_3 + x_3y_1 - x_3y_2 - x_2y_1 - x_1y_3}, \\ \frac{D_2}{D} &= \frac{-(xy_1 + x_1y_3 + x_3y - x_3y_1 - x_1y - xy_3)}{x_1y_2 + x_2y_3 + x_3y_1 - x_3y_2 - x_2y_1 - x_1y_3}, \text{ and} \\ \frac{D_3}{D} &= \frac{xy_1 + x_1y_2 + x_2y - x_2y_1 - x_1y - xy_2}{x_1y_2 + x_2y_3 + x_3y_1 - x_3y_2 - x_2y_1 - x_1y_3}. \end{aligned} \quad (39)$$

If you carry out the multiplications indicated in Equations 34, you get the ratios in Equations 39. This proves the conjecture. The relative weights derived from the ratio of cross-products (i.e., areas of triangles, Equations 34) are the same as the relative weights derived from the equation of the plane (Equations 39).

In conclusion, the z -value within the triangle is the weighted average of the values at the corner (Equation 32), and the relative weights can be calculated by either of two sets of ratios of determinants:

$$\omega_1 = \frac{\begin{vmatrix} x_2 - x & y_2 - y \\ x_3 - x & y_3 - y \end{vmatrix}}{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix}} = \frac{\begin{vmatrix} x & y & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}}{\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}}, \quad (40)$$

$$\omega_2 = \frac{\begin{vmatrix} x_3 - x & y_3 - y \\ x_1 - x & y_1 - y \end{vmatrix}}{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix}} = \frac{-\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}}{\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}}, \text{ and} \quad (41)$$

$$\omega_1 = \frac{\begin{vmatrix} x_1 - x & y_1 - y \\ x_2 - x & y_2 - y \end{vmatrix}}{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix}} = \frac{\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}}{\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}}. \quad (42)$$

The first set of ratios (the column with 2×2 determinants) comes from calculating areas of triangles. The second set (the column with 3×3 determinants) comes from the equation of a plane.

Looking Back

According to our algebra, we have two ways of calculating the same thing. We can check our algebra by using both techniques to solve the same exercise.

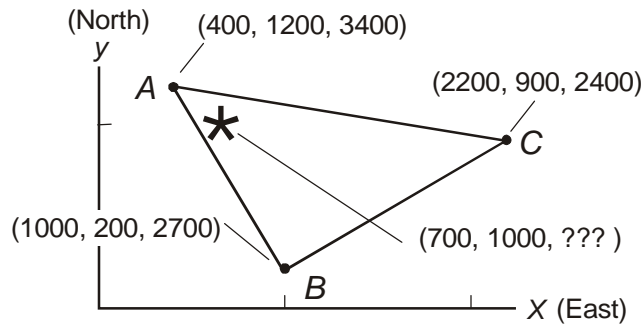


Figure 11. What is the elevation at the asterisk, which is not at the center, an edge, or corner of the triangle?

A sample exercise is shown in Figure 11. The xyz -values at the corners are the same as in Figures 1-3. The asterisk marks the xy -location of an interior location. What do our two methods produce for the z -value at this interior location?

The spreadsheet in Figure 12 finds an answer by both techniques. Down to Row 32, this spreadsheet is the same as the one solving the three-point problem in CG-12 (CG-12, Fig. 11), except that Row 10 has been added for the interior point, and Cell H32 has been added for the calculated z -value at the interior point. The value at H32 is calculated from the slopes-and-intercept form of the equation of the plane (CG-12, Equation 48),

$$z = z_0 + m_x x + m_y y, \tag{43}$$

and the values in cells D30-D32. The same result is produced at H53, which comes from the weighted average and areas of the distal triangles discussed in this essay.

Point Estimation in Geostatistics

An Introduction to Applied Geostatistics by E.H. Isaaks and R.M. Srivastava (Oxford University Press, 1989, 561 pp) is a delightfully clearly written book explaining statistical tools used to analyze spatial data. Except for notation, Equation 11.6 of that book is the same as Equation 17 of this essay. Whereas Equation 17 uses the letter z for elevations at the corners and interior point of the triangle, Equation 11.6 of *Applied Geostatistics* uses the letter v , representing a quantity that has units of ppm.

Referring to Equation 11.6, the authors say (p. 255), "Our triangulation estimate, therefore, is a weighted linear combination in which each value is weighted according to the area of the opposite triangle. This weighting agrees with the common intuition that closer points should receive greater weights". This material is included under the heading "Triangulation" in their

chapter titled "Point Estimation." Triangulation is one of four methods of point estimation methods discussed in that chapter.

	B	C	D	E	F	G	H	I	
3	THE THREE-POINT ESTIMATION PROBLEM IN FIGURE 11								
4									
5	I. Coordinates of the three corners and interior point								
6			x	y	z				
7	A		400	1200	3400				
8	B		1000	200	2700				
9	C		2200	900	2400				
10	D		700	1000	???				
11									
12	II. Linear coefficients for the equation of the plane								
13		a =	1200	3400	1	=	790000		
14			200	2700	1				
15			900	2400	1				
16									
17		b = -	400	3400	1	=	-660000		
18			1000	2700	1				
19			2200	2400	1				
20									
21		c =	400	1200	1	=	1620000		
22			1000	200	1				
23			2200	900	1				
24									
25		d = -	400	1200	3400	=	-5032000000		
26			1000	200	2700				
27			2200	900	2400				
28									
29	III. Slopes and intercept for the equation of a plane								
30		z0=	3106.173						
31		mx=	-0.487654						
32		my=	0.407407					THEN zD=	3172
33									
34	IV. Areas of triangles (Equations 29, 31)								
35									
36	total	A-B-C	600	-1000	=	810,000			
37			1800	-300					
38									
39	opp. A:	B-D-C	300	-800	=	585,000			
40			1500	-100					
41									
42	opp. B:	A-D-C	1500	-100	=	135,000			
43			-300	200					
44									
45	opp. C:	A-D-B	-300	200	=	90,000			
46			300	-800					
47									
48		Sum of the three internal triangles				=	810,000		
49									
50	V. Relative weights (Equations 33)								
51		A	0.722						
52		B	0.167						
53		C	0.111					THEN zD=	3172

Figure 12. Spreadsheet solving the problem posed in Figure 11.

Point estimation, as the term suggests, is the estimation of a measurable quantity at a point where it has not been measured, and it arises often in interpreting map information. Suppose, for example, that you have water-quality data at many wells scattered across a geographic area. What can you say about the water quality at specific locations where you do not have any wells? One statistical way is to use the weighted average of data from three nearby wells (i.e., Equation 17 here, or Equation 11.6 of *Applied Geostatistics*). This technique is called triangulation in

geostatistics for the obvious reason that it uses three data points; there are other techniques that take account of more than three points and use other weighting schemes

The weighted averaging of triangulation in geostatistics is linear interpolation on a plane. As we have shown, it is a variation of the theme of solving the three-point problem familiar from structural geology and hydraulic gradients.

When you hand contour well data by figuring how contours cross the lines between a pair of wells by dividing the line into proportionate parts, you are applying a two-point version of the same equations used in geostatistical triangulation. This is the familiar linear interpolation on a line from school algebra.

Contouring is a form of geostatistical point estimation. In its simplest form, it is akin to the familiar three-point problem.

Binary and Ternary Mixtures

Principles and Applications of Geochemistry by G. Faure (2nd edition, Prentice Hall, 1998, 600 pp) is the textbook of choice for many courses in general geochemistry. Chapter 18, "Mixing and Dilution," gives a useful account of mixtures of two and three end-members (binary and ternary mixtures, respectively). Much of the chapter discusses interpretation of XY -scatterplots, where the random variables X and Y are concentrations of chemical constituents in a collection of, say, water samples (or, in the language of sampling, a sample consisting of individuals – individual specimens – where each specimen is represented by a data point). If the sample is drawn from a population formed by binary or ternary mixing, how does the sample appear on the scatterplot?

A sample from a binary mixture scatters along a straight line on an XY -plot, if the two chemical species are conservative, meaning that they do not react with each other or with the environment as a result of the mixing. The straight line is called a *mixing line*, an important term in the language of geochemistry. A generic example is shown in Figure 13. In the example discussed by Faure, the sample is taken from a channel where water from one lake mingles with water from another. In Faure's example, X is concentration of Sr in $\mu\text{g/L}$, Y is concentration of Ca in mg/L , and P_1 and P_2 are waters of the two lakes (Lake Superior and Lake Huron, respectively). Addition of small amounts of a third component (such as dilution by rainwater) and seasonal variation in the end-members contribute to scatter around the mixing line.

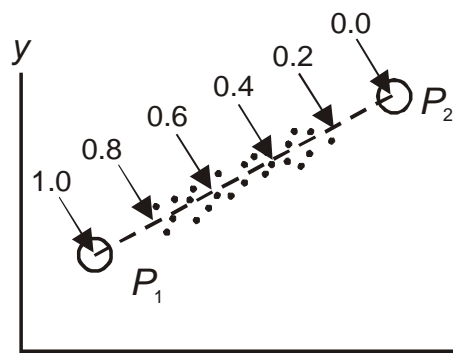


Figure 13. A mixing line connecting P_1 and P_2 . Mixtures range from $f = 1$ for a mixture consisting 100% of P_1 to $f = 0$ for a mixture consisting of 0% of P_1 .

The numbers on Figure 13 indicate the value of f , the fraction of P_1 that is in the mixture. The values of X and Y along the mixing line are clearly the weighted average of the X - and Y -values in P_1 and P_2 , where the relative weights are f and $1-f$, respectively. In other words, the variation of X and Y is described by Equation 12, which is the same, except for notation, as Equation 18.2 of Faure's book. To calculate the fraction of the mixture that is derived from P_1 , one can use the first of Equations 11, which is the same except for notation as Equation 18.6 of Faure's book.

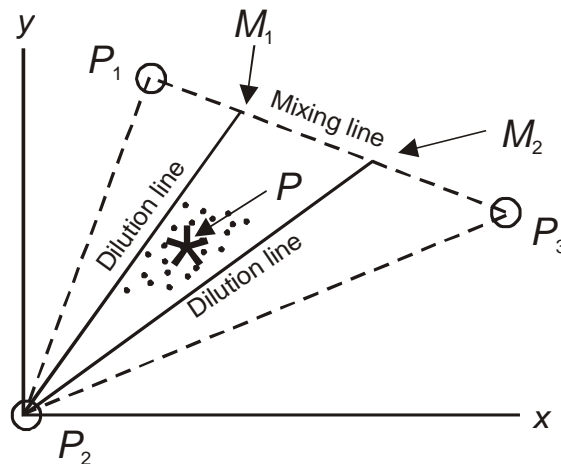


Figure 14. A mixing triangle connecting P_1 , P_2 and P_3 . Mixtures ranging from M_1 to M_2 along the mixing line connecting P_1 and P_3 are diluted with P_2 . The percentage of any end-member in the ternary mixture can be calculated from the area of appropriate triangles and a spreadsheet like that of Figure 12.

In the same way that a sample from a binary mixture is scattered along a mixing line, a sample from a ternary mixture is scattered within a *mixing triangle*. Mixing of three components can occur, for example, as a result of a sequence of two mixing processes. One case discussed by Faure is a solution formed, first, by mixing of two subsurface brines and, then, dilution of that binary mixture with meteoric water.

A generic example is shown in Figure 14. P_1 and P_3 are the high-concentration brines, which form a mixture ranging from M_1 and M_2 on a mixing line. P_2 is meteoric water, which is low enough in dissolved constituents that it plots at the origin. Dilution of the P_1P_3 brine mixture results in an infinite set of mixing lines extending from the M_1 - M_2 line segment to P_3 . The data points from the ternary mixture plot along those mixing (or dilution) lines. The example discussed by Faure is from a study of oilfield brines. X is Sr in 10^2 mg/kg and Y is Na in 10^4 m/kg.

The values of X and Y within the mixing triangle are the weighted average of the X - and Y -values in P_1 , P_2 and P_3 . The weighted average is described by an equation like Equation 17, in which the weights are the fraction of the end-member in the mixture. For any particular mixture, such as P (the sample average) in Figure 14, the fraction of the various end-members in the mixture can be found from the ratios in Equations 40-42. For example, the fraction of P_1 in P is the ratio of the area of triangle P_2PP_3 to the area of triangle $P_2P_1P_3$ and can be found from

Equation 40. Similarly, the fraction of M_1 in P , within a M_1 - M_2 - P_2 mixture, is the ratio of the area of triangle P_2PM_2 to the area of triangle $P_2M_1M_2$ and can also be found from Equation 40.

As an exercise in geometry, you might like to convince yourself that the fraction of the blend that comes from the meteoric-water end-member is the same regardless of which mixing triangle you use. That is, show that

$$\frac{A_{P_1PP_3}}{A_{P_1P_2P_3}} = \frac{A_{M_1PM_2}}{A_{M_1P_2M_2}}. \quad (44)$$

Concluding Remarks

There is much discussion in science education now about interdisciplinary, or integrative, science. Courses are being developed that cross disciplinary boundaries. As the language and problem solver of science, mathematics crosses disciplinary boundaries.

A single mathematical problem can arise in many different contexts and can produce an array of different problems. They may have different languages reflecting the various specialized settings. But they are all simply variations on a single theme. For example, within the single field of geology, variations of the three-point problem include finding the strike and dip of an inclined surface, finding the hydraulic gradient from well data, estimating the value of a measured quantity at a location where it has not been measured, and interpreting geochemical data sampled from ternary mixtures. Whether the problem is attacked graphically, from the equation of a plane, by means of a weighted average and vector products, or as a case of three-dimensional linear interpolation – the problem is still basically the same mathematically.

Any effort to break down the disciplinary boundaries of science in education would be greatly enhanced by developing – as opposed to avoiding – mathematics. The same is true, of course, for breaking down subdisciplinary boundaries within geoscience education.

Acknowledgments

I thank Nicole Elko for introducing me to the geostatistics book by Isaaks and Srivastava, and Thomas Pichler for introducing me to the geochemistry book by Faure.

Appendix: Manipulating Equations

The second edition of the superb little book *Mathematics, A Simple Tool for Geologists* by David Waltham (2nd edition, Blackwell Science, Oxford, 2000, 201pp) is now available. Chapter 3, "Equations and How to Manipulate Them," reviews the basic rules of school algebra.

Deriving Equation 8 from Equation 7 provides a good illustration of the rules of Sections 3.2 and 3.3, "Combining and simplifying equations," and "Manipulating expressions containing brackets," respectively.

Thus, starting with Equation 7,

$$y_3 = y_1 + \left(\frac{y_2 - y_1}{x_2 - x_1} \right) (x_3 - x_1),$$

multiply out the $(y_2 - y_1)$ factor,

$$y_3 = y_1 - y_1 \left(\frac{x_3 - x_1}{x_2 - x_1} \right) + y_2 \left(\frac{x_3 - x_1}{x_2 - x_1} \right).$$

Then factor out y_1 ,

$$y_3 = y_1 \left(1 - \frac{x_3 - x_1}{x_2 - x_1} \right) + y_2 \left(\frac{x_3 - x_1}{x_2 - x_1} \right),$$

and substitute for the 1 in the first set of parentheses,

$$y_3 = y_1 \left(\frac{x_2 - x_1}{x_2 - x_1} - \frac{x_3 - x_1}{x_2 - x_1} \right) + y_2 \left(\frac{x_3 - x_1}{x_2 - x_1} \right).$$

Finally, combine the terms in the first set of parentheses to get

$$y_3 = y_1 \left(\frac{x_2 - x_3}{x_2 - x_1} \right) + y_2 \left(\frac{x_3 - x_1}{x_2 - x_1} \right),$$

which is Equation 8.

The algebra between Equations 7 and 8 is the kind of material that is invariably left out of specialized geology textbooks. To include it would make the textbooks simply too long. Many beginning graduate students encountering mathematics-intensive geology courses for the first time, however, find themselves unable to recreate the missing steps when they work through these books. Such students would be well served to have *Mathematics, A Simple Tool* in their personal library and to work through it for review.