

Computational Geology 10

The Algebra of Deduction

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Introduction

Computational Geology 7 "The Algebra of Unit Conversions" (September 1999) called attention to one of the giant steps in the development of mathematics: the replacement of numbers by symbols such as x , y and z . This invention of symbolic algebra by François Viète (1540-1603) was one of the breakthroughs of the Renaissance. With symbolic algebra, equations could be manipulated to solve problems. It was no longer necessary to use the ancient, *ad hoc* algorithms of the Islamic scholars.

This column discusses another enormous step in the history of algebra. In this algebra, symbols do not represent quantities, but rather words or sentences. Expressions containing these symbols can be manipulated to deduce the consequences of prior expressions or to evaluate the validity of arguments. This algebra is taught in freshman courses on logic and critical thinking. It informs the *-logy* of geology, biology, and other sciences.

The story of symbolic logic begins with Gottfried Wilhelm Leibniz (1646-1716), a universal genius and co-inventor of the infinitesimal calculus (see CG-5 "If Geology then Calculus", March, 1999). A diplomat and political advisor by profession, Leibniz was "what we would call in this day and age a jet-setter" (Solomon and Higgins (1996, p.190). In his spare time, he was a well-known philosopher (Russell, 1937; Durant and Durant, 1963, Chap. 23) and mathematician (Hofman, 1974; Bell, 1937, Chap. 7; Hollingdale, 1991, Chap. 11). "He thought incessantly. His unrelenting curiosity was attracted to everything...." (Bell, 1945, p. 149). According to Edwards (1979, p. 231), "the breadth of his fundamental contributions ... is probably not matched by the work of any subsequent scholar."

Before getting to Leibniz and logic, it is worth noting that Leibniz was an early geologist. According to the *Encyclopedia Britannica Online* ("Leibniz, Gottfried Wilhelm", http://www.eb.com:180/bol/topic?artcl=47669&seq_nbr=1&page=n&isctn=2 accessed 14 Dec 1999).

.... among the great philosophers of his time, he was the only one who had to earn a living. As a result, he was always a jack-of-all-trades to royalty.

Trying to make himself useful in all ways, Leibniz proposed that education be made more practical, that academies be founded; he worked on hydraulic presses, windmills, lamps, submarines, clocks, and a wide variety of mechanical devices; he devised a means of perfecting carriages and experimented with phosphorus. He also developed a water pump run by windmills, which ameliorated the exploitation of the mines of the Harz Mountains, and he worked

in these mines as an engineer frequently from 1680-1685. Leibniz is considered to be among the creators of geology because of the observations he compiled there, including the hypothesis that the Earth was at first molten....

Leibniz in 1685 was the historian for the House of Brunswick. He was commissioned to write the history of the family to “prove, by means of genealogy, that the princely house had its origins in the House of Este, an Italian princely family” (*Encyclopedia Britannica Online*). Leibniz began the history (Rudwick, 1976, p. 91):

...with what he subtitled “A dissertation on the original form of the Earth and on the vestiges of its most ancient history in the very monuments of nature”.... Adopting a Cartesian explanation of the origin of the Earth as an incandescent globe, Leibniz postulated the consolidation of an original crust, the condensation of an initially universal ocean, and the subsequent deposition of a sequence of strata containing fossils, with the simultaneous diminution of the ocean by evaporation. The bulk of the essay was in fact devoted to the description and illustration of fossils, and to the demonstration of their organic origin, as a crucial part of his whole synthesis.

Leibniz’s *Protogaea* was published posthumously in 1749. According to Rudwick (1976), it was highly influential in a number of ways. It conformed to both scripture and reason. It provided a model of Earth-history that allowed for the organic origin of fossils. It preserved emerging ideas that strata are sequential deposits (notably the ideas of Steno [Niels Stenson, 1638-1686, known now for the Law of Superposition], and John Woodward [1665-1728, British physician whose *Essay toward a Natural History of the Earth* of 1695 interpreted the origin of fossiliferous strata of England]). Most importantly, it made it “possible for the different fossils embedded in successive strata to become evidence of the history of life itself, although that conclusion was not at first drawn in any detail” (Rudwick, 1976, p. 91).

Leibniz's Vision

"Mentioning the name of Leibniz is like referring to a sun rising." With that sentence, Scholz (1961, p. 50) started the chapter entitled "The Modern Type of Formal Logic" in his *Concise History of Logic*. Scholz was referring to Leibniz's interest in developing a language for calculating logical conclusions.

Before Leibniz, logic took the form of syllogisms composed of words. A syllogism consists of three statements. The first two are premises, and the third is a conclusion that is supposed to follow from the premises. A familiar type of syllogism is immortalized by:

Premise: All men are mortal.
Premise: Socrates is a man.
Conclusion: Socrates is mortal.

This type of syllogism, in which the statements refer to membership in classes (e.g., the class of mortals; the class of men), is characteristic of Aristotle's logic. It is an example of *predicate logic*.

A second type of syllogism can be illustrated by:

Premise: If it is not day, it is night.

Premise: It is not day.

Conclusion: It is night.

This type of syllogism traces back to the early Stoics, most notably Chrysippus (~280-~205 B.C.) and exemplifies *propositional logic*. Chrysippus recognized five basic valid constructions (Kneale and Kneale, 1964):

1. If the first, then the second; but the first; therefore the second.
2. If the first, then the second; but not the second; therefore not the first.
3. Not both the first and the second; but the first; therefore not the second.
4. Either the first or the second; but not the first; therefore the second.
5. Either the first or the second; but not the second; therefore the first.

The syllogism concerning day and night is an example of the first of these constructions.

These two types of syllogism became the mode of medieval thinking called Scholasticism. Following St. Augustine (354-430), the Scholastics of ca. 1050-1350 believed that "the same God who was revealed through Scripture had given human beings the faculty of reason, which enabled them to come to know the truth" (Solomon and Higgins, 1996, p. 143).

Following the Middle Ages, there was the Renaissance in Europe, the Reformation, and the Scientific Revolution. As science overcame the authority of the church, Europe entered the Age of Reason (Solomon and Higgins, 1996, p. 192):

(F)ollowing Descartes and the new science, the Enlightenment philosophers put great trust in their own ability to reason, in their own experience and their own intellectual autonomy.... Through reason, they believed, they would not only tap the basic secrets of nature through science but would also establish a living paradise on earth....

In 1666, about 30 years after Descartes' *Discourse on Method* (1637), Leibniz presented a dissertation for a doctorate of law at the University of Leibzig. The dissertation was entitled *De Arte Combinatoria* ("On the Art of Combination"). He was 20 years old at the time.

Leibniz was turned down at Leibzig on account of his youth (Durant and Durant, 1963). Undeterred, he moved to Nuremberg and promptly submitted a new dissertation to the University of Altdorf; he was awarded not only a doctorate but also an offer of the University's Chair of Law. He declined, saying he had "very different things in view" (Hollingdale, 1991, p. 254). Thus Leibniz began a career in service to royalty (Appendix 1). Nevertheless, *De arte combinatorie* began a theme that persisted throughout his life.

The thesis of *De arte combinatorie* was that all reasoning and discovery can be reduced to combinations of elements such as numbers or words. In this vein, Leibniz sought a *characteristica generalis*, a general symbolic language into which all processes of rational human thought could be translated. He believed such a language "would eliminate the mental labor of routine and repetitive steps. His goal was the creation of a system of notation and terminology that would codify and simplify the essential elements

of logical reasoning.... Such a universal 'characteristic' or language, he hoped, would provide all educated people -- not just the fortunate few -- with the powers of clear and correct reasoning" (Edwards, 1979, p. 232).

The crucial idea of Leibniz's symbolic language was to extract the reasoning from the words. This separation would allow one to focus on relationships and not be misled by the meaning of the words. The analog is doing algebra with symbols (e.g., x , y) while not thinking of the numbers that the symbols represent. Here is how Scholz (1961, p. 53) described the analogy:

And what is it that we expect from such a [symbolic] logic? We require of it that it will render syllogizing just as independent of thinking or the meaning content of the propositions involved in the syllogism, as modern mathematics has made calculating in the widest sense of the word right down to the magnificent feats of the modern infinitesimal calculus independent of thinking of the meaning content of the symbols involved in the calculation.

With the eye of genius Leibniz saw that the unparalleled advance of modern mathematics rests upon this unburdening of thought. Relieving thought in this way tremendously facilitates reasoning. Syllogizing is thus freed of all sorts of unnecessary thought operations by virtue of ingenious substitutions. At the same time, syllogizing is exemplarily insured against errors to which content-centered thinking is constantly prone.

If logic could be done with symbols – devoid of meaning – then logical conclusions could be derived by calculation. This means that deduction would become a *calculus*. Today, the two types of logic are often called the *propositional calculus* and the *predicate calculus*.

To geology students, the word calculus, of course, means the *infinitesimal calculus* of rates and sums. As noted in CG-5, Leibniz's notation for the derivative, dy/dx , and the integral, $\int ydx$, facilitates calculation and problem solving. According to Edwards (1979, p. 232), Leibniz's infinitesimal calculus

... is the supreme example, in all of science and mathematics, of a system of notation and terminology so perfectly mated with its subject as to faithfully mirror the basic logical operations and processes of that subject. It is hardly an exaggeration to say that the calculus of Leibniz brings within the range of an ordinary student problems that once required the ingenuity of an Archimedes or a Newton.

The now-familiar dy/dx and $\int ydx$, however, was only part of what Leibniz had in mind for using symbols to facilitate calculation.

Mathematics and Logic

Although Leibniz is widely regarded as the first mathematical logician, he did not influence the development of the subject. His work on symbolic logic was not published until the beginning of the twentieth century.

George Boole (1815-1864) is generally considered to have started the first continuous development of mathematical logic with publication of *The Mathematical*

Analysis of Logic (1847). In the introduction to that work, Boole clearly stated the fundamental principle that he wanted to transfer from symbolic algebra to logic (Bochenski, 1961, p. 278):

They who are acquainted with the present state of the theory of Symbolical Algebra are aware that the validity of the processes of analysis does not depend upon the interpretation of the symbols which are employed, but solely on the laws of their combination.

But, he said, the symbolic algebra had been applied to quantities and, mistakenly, taken to be applicable only to quantities. A similar calculus could be developed for logic (Bochenski, 1961, p. 279):

That to the existing forms of analysis a quantitative interpretation is assigned, is the result of the circumstances by which those forms were determined, and is not to be construed into a universal condition of Analysis. It is upon the foundation of this general principle, that I purpose to establish the Calculus of Logic, and that I claim for it a place among the acknowledged forms of Mathematical Analysis...

Unlike Leibniz and others who envisaged extracting a calculus from processes of rational thought, Boole intended to devise "a formal construction for which an interpretation is sought only subsequently" (Bochenski, 1961, p. 279). Indeed, interpretations of Boole's algebra of reason have been applied to both propositional and predicate logic. The latter interpretation – which became an algebra of sets (Boolean algebra) – is best known.

The formal construction that has evolved into the symbolic logic taught in colleges today was developed by Gottlob Frege (1848-1925), considered the founder of mathematical logic. Frege's epochal work, published in 1879, was *Begriffsschrift*, meaning "concept script", a translation of Leibniz's *characteristica generalis*. Whereas Boole endeavored to make logic a part of mathematics, Frege viewed mathematics as an application of logic. His final work, *Grundgesetze der Arithmetik (The Basic Laws of Arithmetic)*, 1893) stated the position explicitly by stating its purpose as "derivation of the simplest laws of Numbers by logical means alone" (Frege, 1964, p. 29).

Frege's work was the start of logicism, the idea that all of mathematics could be derived from the principles of logic. This line of thinking, which endeavored to establish the fundamental principles of mathematics, is characterized by very detailed, rigorous logical proofs. It culminated in the work of Bertrand Russell (1872-1970), in his *Principles of Mathematics* (1903) and the monumental three volumes of *Principia Mathematica* (1910-1913) by Alfred North Whitehead (1861-1947) and Russell. The logic in this work is basically that of Frege, with a notation developed by Giuseppe Peano (1858-1932). The symbolic calculus taught in freshman logic courses is, in its essentials, that in the *Principia*.

Requirements of deduction

Frege's project to "derive the simplest laws of Numbers by logical means alone" required three things (Furth, in Frege, 1964, p. v-vi):

1. A set of assertions representing truths of logic (i.e., logically true premises)
2. A set of principles that "would be indisputably recognized as sound rules of inference, in the sense that, applied to logically true premises, the transformations permitted by these principles could not issue in any but logically true conclusions."
3. Derivations of the propositions of arithmetic from (1) by means of (2).

The third item is of interest to the relationship between mathematics and logic. Frege's work on the first two, however, was more fundamental because it established the framework of a logical system and clarified what it means to say that something is rigorously proved. Thus, quoting Frege's *Grundgesetzen* (Frege, 1964, p. 29):

... considerably higher demands must be made on the conduct of proof than is customary in arithmetic. A few methods of inference must be marked out in advance, and no step may be taken that is not in accordance with one of these. Thus in passing on to a new judgment, one must not be satisfied, as the mathematicians have nearly always been hitherto, with the transition's being evidently correct; rather, one must split it into its simple logical steps of which it is composed.... In this way, no presupposition can pass unnoticed; every axiom must be uncovered. It is indeed precisely the presuppositions made tacitly and without clear awareness that obstruct our insight of the epistemological nature of a law.

Later (1896), Frege put the point in a way that it should get our attention, even though we may not be particularly interested in the logical underpinnings of mathematics (Bochenski, 1961, p. 293):

"Words such as 'therefore', 'consequently', 'since' suggest indeed that inference has been made, but say nothing of the principle in accordance with which it has been made.... In an inquiry which I here have in view, the question is not only whether one is convinced of the truth of the conclusion...; one must also bring to consciousness the reasons for this conviction.... Fixed lines on which the deductions must move are necessary for this, and such are not provided in ordinary language.

In the technical literature of geology, when do you ever see an explanation of what the author means by "therefore" or "consequently"? When does the author of the paper ever tell you what principle justifies the claim that the conclusion follows from the premises?

Frege's functions

What did Frege do that would provide the "fixed lines on which deductions must move"? He brilliantly expanded the concept of a mathematical *function*. His functions led to a way of validating conclusions by calculation.

Before the time of Frege, the concept of a function was limited to relations between quantities, and formulas stating those relations. For example, a classic definition

of function – that of Euler's *Introductio in analysin infinitorum* of 1748 (see CG-9, "The Exponential Function") – was (Edwards, 1979, p. 271):

A function of a variable quantity is an analytic expression composed in any way whatsoever of the variable quantity and numbers or constant of quantities.

This is the function we have in mind when we say, for example, that $f(x) = x^2 + 1$. In such a function, x represents quantities (the independent variable), and $x^2 + 1$ is the analytic expression. The function produces *quantities* (the dependent variable).

Frege had something much larger in mind (Frege, 1964, p. 33-37). To start, Frege distinguished functions from objects. Functions are *unsaturated* in that they contain gaps as in the empty space of $f()$. Objects, by contrast, are *saturated* – fully designated. The gaps in a function are filled with *arguments* (one of his words that have come down to us). A single object can have many proper names: for example 4, 2^2 and $8/2$ are all names of the same object. A function is no longer a function when its argument is specified by a proper name; it becomes a *value*, which is a type of object. Functions can have more than one argument. When all the arguments are specified by proper names, the result is, again, a value.

Frege included *equations as functions*. When arguments of this type of function are specified, the result (value) is what he called a *truth-value* (another all-important word that has come down to us). Thus when arguments are substituted that make one side of the equation equal to the other, the truth-value is T (what he called "*the True*"). When arguments are substituted that make one side of the equation larger than the other, the truth-value is F (his "*the False*"). Thus for any function that is an equation, all the possible substitutions can produce only two values: T and F. The same is the case for any function that is an inequality.

Frege's definition of function allows any type of object to be an argument of appropriately constructed functions. The only requirement is that the function, upon saturation, must produce an object (value). The innovation that was epochal for symbolic logic was that he defined functions that used *truth-values as arguments*. Such *truth-functions*, as they are called now, *produce truth-values*.

Frege's way of expressing truth-functions was diagrammatic (and cumbersome), and there is no point in going into it here. But, to follow Frege's thinking, consider the following three truth-functions: $\Phi(\xi)$, $\psi(\xi, \zeta)$, and $\Gamma(\xi, \zeta)$. The first function contains one argument; the others each contain two arguments. The arguments can be filled only with truth-values (T and F). The objects produced by the functions can similarly be only truth-values (T and F). In more-modern language, these functions map a domain consisting of combinations of truth-values onto a range of truth-values. That is what it means to say that a function is a truth-function.

Now, define the truth-function $\Phi(\xi)$ so that it produces the truth-value that is the *opposite* of its argument. In other words, $\Phi(T) = F$, and $\Phi(F) = T$. [Said another way, let $\Phi(\xi)$ be F for $\xi = T$, and let $\Phi(\xi)$ be T for $\xi = F$.]

Secondly, define the truth-function $\psi(\xi, \zeta)$ so that it produces F for all combinations of values of ξ and ζ , *except* for the case that $\xi = T$ and $\zeta = T$, in which case $\psi(\xi, \zeta)$ is T. In other words: $\psi(T, T) = T$; $\psi(T, F) = F$; $\psi(F, T) = F$; and $\psi(F, F) = F$.

And finally, define a third truth-function $\neg(\xi, \zeta)$ so that it produces T for all combinations of values of ξ and ζ , *except* for the case that $\xi = T$ and $\zeta = F$, in which case $\neg(\xi, \zeta)$ is F. In other words: $\neg(T, T) = T$; $\neg(T, F) = F$; $\neg(F, T) = T$; and $\neg(F, F) = T$.

Truth-functions of propositional logic today have a notation that is decidedly different than either Frege's or the one that I have just given to illustrate the parallel between truth-functions and the familiar $f(x)$ or $g(x, y)$. In the notation that is actually used in symbolic logic, arguments are indicated by lower-class letters such as p , q , r , and s . These arguments are not enclosed within parentheses attached to the symbol representing the function. Rather, the arguments are strung together – concatenated – by symbols acting as connectors. For example, the truth-function $\neg(\xi, \zeta)$ is represented by $p \wedge q$ (or, in some versions, $p \bullet q$) where p is used for ξ and q for ζ . Similarly, the truth-function $\neg(\xi, \zeta)$ is represented by $p \rightarrow q$ (or $p \supset q$), and $\neg(\xi)$, the one-argument truth-function, is represented by $\sim p$.

In addition to these three truth-functions, there are two more: $p \vee q$ and $p \leftrightarrow q$. For the first, the function $(p \vee q)$ is defined to be T for all combinations of truth-values of p and q , except for the case that p is F and q is F, in which case $p \vee q$ is F. For the second, the function $(p \leftrightarrow q)$ is defined to be T if the truth-values of p and q are the same, and F if the truth values of p and q are different.

You may have noticed that defining truth-functions in sentences like I have just done produces sentences that might need to be read a few times. It is much more effective to use a table to communicate these definitions. Such truth-tables were introduced by one of the leading philosophers of the last (20th) century: Ludwig Wittgenstein (1899-1951). These are routinely taught in freshman logic courses. The definitions of the five key truth-functions (functions I-V) are listed in Tables 1 and 2. Compare them to the definitions given in the preceding paragraphs.

	I
p	$\sim p$
T	F
F	T

Table 1: One-argument truth-function

		II	III	IV	V
p	q	$p \wedge q$	$p \vee q$	$p \rightarrow q$	$p \leftrightarrow q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	F	T	T	F
F	F	F	F	T	T

Table 2. Two-argument truth functions

Calculating more truth-functions

Functions I-V form the core of propositional calculus. The arguments (p , q , etc.) of the functions represent *propositions*. Propositions are assertions (sentences). More

specifically, a proposition is a statement that has truth-value; i.e., *it is the kind of statement that can be true or false*. (Whether or not you know the statement is true or false is irrelevant; the requirement is that it must be the kind of statement that, in concept, can be true or can be false).

The truth-functions make *elementary* propositions (p, q , etc.) into *compound* propositions ($\sim p, p \wedge q, p \vee q$, etc.) The five basic truth-functions all have names and are translatable into words. Function I, $\sim p$, is called *negation* and is read "not p ". Function II, $p \wedge q$, is called *conjunction* and is read " p and q ". Function III, $p \vee q$, is *inclusive disjunction* and is read "either p or q or both". Function IV, $p \rightarrow q$, is the *conditional* and is read "if p , then q ". Finally, Function V, $p \leftrightarrow q$, is the *biconditional* and is read " p is equivalent to q ".

Functions are concatenated to form more truth-functions. Five such examples of functions of p are shown in Table 3. The truth-values that any one of them produces can be worked out step by step by building up the function in a truth-table. For example, the third function of Table 3 ($p \wedge \sim p$) follows from the first two columns ($p, \sim p$) and the definition of the conjunction (Function II, Table 2). Similarly, the next function of Table 3, $\sim (p \wedge \sim p)$, is simply the negation of the column that precedes it.

		1	2	3	4	5
p	$\sim p$	$\sim \sim p$	$p \vee \sim p$	$p \wedge \sim p$	$\sim (p \wedge \sim p)$	$p \leftrightarrow \sim \sim p$
T	F	T	T	F	T	T
F	T	F	T	F	T	T

Table 3. Some calculated functions of p .

It should be noted that these derived, concatenated functions are simply functions of functions. This can be illustrated by using the notation of Greek letters introduced earlier for three of the basic truth-functions. For example, function 3 of Table 3 would be $\psi(\xi, \Phi(\xi))$. Its negation (function 4) would be $\Phi(\psi(\xi, \Phi(\xi)))$. The advantage of using the concatenated expressions of Table 3 is obvious.

Functions 2 and 4 of Table 3 have special significance in logic. Function 2, which says, "*you either have p or you don't have p* ", is the Law of the Excluded Middle. Function 4, which says, "*you cannot have both p and not p* ", is the Law of Contradiction. These laws are fundamental and date back to Aristotle. They are another way of stating the meaning of the term *proposition*, the subject of propositional logic.

You might note that column 3 of Table 3 (i.e., function 1, $\sim \sim p$) is the same as column 1 (p). Whenever you have two columns that are the same, the functions that the columns display can be connected by an equivalence sign (\leftrightarrow) to define a new function which will be T in all cases (all rows). Such is the case in the last column. Function 5, $p \leftrightarrow \sim \sim p$, is the Law of Double Negation.

Five more calculated functions are shown in Table 4. These functions use two elementary propositions, p and q . These examples all involve the conditional ($p \rightarrow q$, Function IV of Table 2). The conditional lies at the heart of argumentation.

				IV	6	7	8	9	10
p	q	$\sim p$	$\sim q$	$p \rightarrow q$	$q \rightarrow p$	$\sim q \rightarrow \sim p$	$\sim(p \wedge \sim q)$	$(p \rightarrow q) \wedge (q \rightarrow p)$	$(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$
T	T	F	F	T	T	T	T	T	T
T	F	F	T	F	T	F	F	F	T
F	T	T	F	T	F	T	T	F	T
F	F	T	T	T	T	T	T	T	T

Table 4. Some calculated functions of p and q .

It is useful to know some terminology concerning conditionals. Recall, $p \rightarrow q$ is translated "if p , then q ". The "if-proposition" (p) is the *antecedent*; the "then-proposition" is the *consequent*. The antecedent is the *sufficient* condition; the *consequent* is the *necessary* condition. Thus, given $p \rightarrow q$, one can say that p is the *sufficient condition of q* , and q is the *necessary condition of p* . Saying that q is the necessary condition of p is the same as saying that *whenever* you have p you will have q ; there *is no way* of avoiding q , if you have p . Therefore, another translation of $p \rightarrow q$ is " p only if q ".

The conditional formed by reversing the antecedent and consequent (i.e., $q \rightarrow p$) is called the *converse* of $p \rightarrow q$. This is function 6 of Table 4. Comparing the truth-values of function 6 and function IV (Table 4), it is clear that the two are NOT the same function. An important principle of logic is: *The converse does not necessarily follow*. Forgetting this very obvious fact leads to much fallacious reasoning, including some applications of uniformitarianism (next section and next column).

The conditional formed by reversing the *negation* of the antecedent and the *negation* of the consequent (i.e., $\sim q \rightarrow \sim p$) is called the *contrapositive* of $p \rightarrow q$. This is function 7 of Table 4. Comparing the truth functions of function 7 and function IV shows that the two are the same function. In other words, " p gets you q " is the same as "you won't have p if you don't have q ." This principle is extremely useful.

Function 8 is another function that is the same as $p \rightarrow q$. Function 8 says "you can't have p and not have q ". This last statement is usually given in freshman logic to explain the meaning of $p \rightarrow q$.

Function 9 is the conjunction of the conditional ($p \rightarrow q$) and its converse ($q \rightarrow p$). You can see that this function is the same as the biconditional (function V of Table 2). Because $p \rightarrow q$ can be read " p only if q ", and $q \rightarrow p$ can be read " p , if q ", their conjunction can be read " p if and only if q ". Thus saying " p is logically equivalent to q " is the same as saying " p if and only if q ". Putting a few things together, you should also see that $(p \rightarrow q) \wedge (\sim p \rightarrow \sim q)$ is the same as $(p \leftrightarrow q)$.

Function 10 is a biconditional constructed from a conditional and its contrapositive. We have already shown that the truth-values of the conditional and contrapositive are the same (functions IV and 7 of Table 4). So, the truth-values of function 10 are all T (from the definition of function V). Such a function – one that is T for all truth-values of the constituent elementary propositions (p, q) – is called a *tautology* or a *law of logic*. Function 10 is the Law of Contraposition.

A law of logic is a compound proposition that produces a column of Ts in a truth-table. Another way of saying this is that a law of logic is a truth-function that maps a domain consisting of combinations of truth-values onto a range consisting of a single

truth-value, T. Many laws of logic have names (e.g., the Law of Double Negation, the Law of the Excluded Middle, and the Law of Contradiction of Table 3, and the Law of Contraposition of Table 4).

Laws of logic form the basis of valid deduction. They provide the "fixed lines on which deductions must move," quoting Frege again ((Bochenski, 1961, p. 293).

"Therefore"

What do you mean when you say the word *therefore*? According to the quotation from Frege's *Grundgesetzen*, you communicate that you have made an inference. In other words, you have been led to a conclusion by preceding statements (premises). What's more, you communicate that you are convinced that the conclusion is true, because you believe the premises to be true.

The concatenation of premises and conclusion is called an *argument* (this is a different use of "argument", than that, following Frege, referring to the independent variable of a function). In general, one can write an argument as a compound proposition:

$$[(P_1) \wedge (P_2) \wedge (P_3) \wedge \dots] \rightarrow \text{Conclusion} \quad (1)$$

where P_1, P_2 , etc. indicate the premises. In other words, the premises of an argument can be collected into a conjunction that is the antecedent in a conditional that contains the conclusion as its consequent.

There are different types of arguments. *Deductive* arguments, the subject of this column, are unique in that they *can guarantee* the truth of the conclusion. That is, *if the premises are true, and the deduction is done correctly, the conclusion is sure to be true*. No other type of inference can make that claim.

What does it mean to do the deduction correctly? The deduction is done correctly if the *structure* of the argument is a law of logic, a tautology. One sees the structure of the argument when all the elementary propositions are replaced by symbols (p, q etc); thus, pleasing Leibniz, you look at the argument without regard to the meaning of the words represented by the symbols. Then, you ask, "Is the overall conditional [i.e., the arrow in (1)] T for all possible combinations of truth-values of p, q , etc.?" If the answer is "Yes", the argument is crafted in such a way that it is impossible for all the premises to be true and the conclusion false. Such an argument is said to be *valid*.

One can determine whether an argument structure is valid by constructing a truth table. Here are two classic examples:

$$\begin{aligned} & [(p \rightarrow q) \wedge p] \rightarrow q, & (2) \\ \text{and} & [(p \rightarrow q) \wedge q] \rightarrow p. & (3) \end{aligned}$$

In each, the conditional $p \rightarrow q$ is P_1 of (1). In (2), the antecedent in $p \rightarrow q$ is affirmed at P_2 of (1), and the consequent in $p \rightarrow q$ is concluded. In (3), the consequent in $p \rightarrow q$ is affirmed at P_2 , and the antecedent in $p \rightarrow q$ is concluded. Not surprisingly, the two argument structures are called *affirming the antecedent* and *affirming the consequent*, respectively. Truth tables for them are shown in Table 5. In the first (Table 5A), the

final column is T in all cases, and so the argument structure is valid. In the second (Table 5B), the final column is not all T, and so the argument structure is not valid.

Affirming the antecedent is the first of Chrysippus's five valid syllogisms listed at the beginning of this essay. The five are listed in symbolic form in Table 6. Proving that these five syllogisms are indeed valid (i.e., laws of logic) is a good exercise in using truth-tables – i.e., the algebra of propositional calculus.

The propositional calculus is applied through substitution instances. A substitution instance is created when the symbols (p , q , etc.) are replaced by actual propositions -- assertions that you wish to make.

A.

p	q	$p \rightarrow q$	$(p \rightarrow q) \wedge p$	$[(p \rightarrow q) \wedge p] \rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

B.

p	q	$p \rightarrow q$	$(p \rightarrow q) \wedge q$	$[(p \rightarrow q) \wedge q] \rightarrow p$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

Table 5. Truth tables for two classic syllogisms. A. Affirming the antecedent is valid. B. Affirming the consequent is not valid.

1	$[(p \rightarrow q) \wedge p] \rightarrow q$
2	$[(p \rightarrow q) \wedge \sim q] \rightarrow \sim p$
3	$[\sim (p \wedge q) \wedge p] \rightarrow \sim q$
4	$[(p \vee q) \wedge \sim p] \rightarrow q$
5	$[(p \vee q) \wedge \sim q] \rightarrow p$

Table 6. Symbolic notation for the five basic valid constructions of Chrysippus.

Any substitution instance of a valid argument is itself valid, whether or not the conclusion is true. To convince someone that a conclusion is true, you must show that the argument is valid *and* that the premises are true. Such an argument – valid, with true premises – is said to be *sound*. An example of this type of argumentation is illustrated in CG-5, where I tried to convince you that the proposition "If you understand geology, you understand calculus" is true.

The problem with invalid arguments is that true premises do not guarantee a true conclusion. If your argument is not valid, it may be that your premises and conclusion are all true. But you *cannot say* that the conclusion is *demonstrated* to be true by the truth of the premises. The reason is that the same argument structure can be used to

generate a false conclusion from true premises. The point can be illustrated by the following two examples:

Heliocentric world view:

- P_1 . If Copernicus's view (that the axis of the rotating Earth remains parallel to itself as the Earth revolves around the sun) is correct, then there would be seasons with different temperatures, duration of daylight hours, and perceived position of the sun in the sky.
- P_2 . There are such seasons.
- C. (Therefore) Copernicus's view is correct.

Geocentric world view:

- P_1 . If Aristotle's view (that the Earth is stationary at the center of the universe and the sun is carried around the Earth on a celestial sphere with an axis inclined relative to that of the celestial sphere carrying the stars) is correct, then there would be seasons with different temperatures, duration of daylight hours, and position of the sun in the sky.
- P_2 . There are such seasons.
- C. (Therefore) Aristotle's view is correct.

The two arguments both have the structure of affirming the consequent. In both arguments, the premises (P_1 , P_2) are T. *But*: the conclusion is true in the first argument and false in the second. Thus affirming the consequent produces a *true* conclusion from true premises sometimes, and a *false* conclusion from true premises sometimes. Clearly, affirming the consequent is an argument structure that *does not guarantee* a true conclusion from true conclusions. Not surprisingly, affirming the consequent is a classic logical fallacy.

Knowing that the two preceding arguments are substitution instances of the fallacy of affirming the consequent, would you say that the word "Therefore" is appropriate in those two arguments? If not, then what do you think of the "Therefore" in the following argument?

Low-angle cross-bedding:

- P_1 . If sand is deposited on a beach, the sand is characterized by gently dipping cross-beds.
- P_2 . Gently dipping cross-beds characterize some Cretaceous sandstones of Montana.
- Therefore*: Those Cretaceous sandstones in Montana were deposited in the beach environment.

Overall, the argument illustrates an application of uniformitarianism. We can stipulate that all three statements are true. But what about the "therefore"? Is it appropriate? Does the truth of these premises *demonstrate* that the conclusion is true? Do you recognize that the argument is another example of affirming the consequent? (For more, see CG 11.)

Concluding Remarks.

The concept of function [a word, incidentally, introduced by Leibniz (Courant and Robbins, 1941, p. 272)] is one of the most important in mathematics. There is broad agreement in the mathematics education community that the concept of function can be a unifying concept to form a thread through all levels of the mathematics curriculum and to make connections with other disciplines (NCTM, 1989; Day, 1995, Kaput, 1999; Knuth, 2000).

Research in mathematics education, however, has found that students have problems with the concept of function (Markovits and others, 1988, Vinner and Dreyfus, 1989). Specifically, there seems to be a disconnect between the concept that is taught and the concept that is internalized by students. The concept, as taught, is the modern one, that a function is a *rule* that maps elements from one set, called the domain, to another set, called the range. What students visualize is a formula – an analytic expression in the words of Euler.

Historically, the path from a formula to a rule that is not necessarily expressed as a formula involved concepts of continuity, counterexamples, and "pathological functions" (Kleiner, 1989) – subjects which, it seems to me, only a mathematician would appreciate. The concept of truth-function, as used in freshman logic, however, seems more interesting.

The function we call a conditional ($p \rightarrow q$), for example, involves a rule, a domain, and a range. The rule is easily expressed as a table (in Table 3), not an analytical expression. The domain is the set consisting of TT, TF, FT, and FF; the range is the set consisting of T and F. If it is more comfortable to think in terms of numbers, one can think of the domain as consisting of the four points (1,1), (1,0), (0,1), and (0,0) on a plane, and the range consisting of 1 and 0 on a line. The rule is that the conditional function maps ((1,0) onto 0, and (1,1), (0,1), and (0,0) all onto 1.

On the matter of connections between mathematics and other disciplines, such a function is no less relevant to geology, other sciences and thinking in general than are the well-known functions expressed by formulas such as $y = mx + b$, or $y = y_0e^{ax}$, or $y = ax^b$.

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