Restricted Symmetric Permutations

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Permutations and Pattern Avoidance

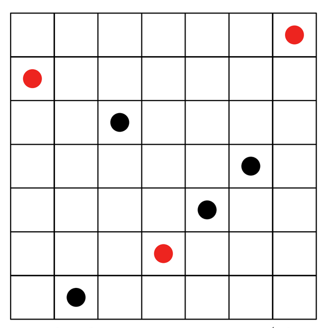
A permutation of $1, 2, \ldots, n$ is a listing of $1, 2, \ldots, n$ in some order. S_n is the set of all permutations of length n.

π, σ are permutations. π contains σ whenever π has a subsequence with the same length and relative order as σ.

π avoids σ whenever π does not contain σ.

The diagram of a permutation π (length n) is an $n \times n$ table where the box $(i, \pi(i))$ is marked with a dot for all $1 \leq i \leq n$.

Example:

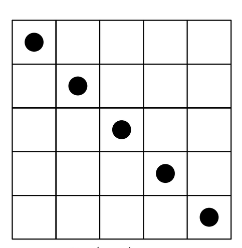


The diagram of 6152347, which contains 213 (shown in red) and avoids 231

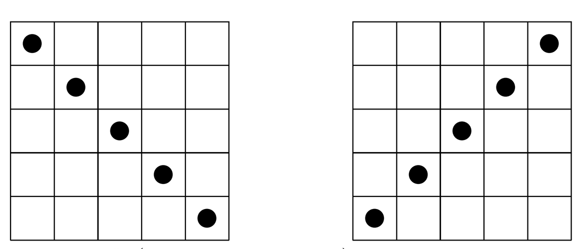
Let R be a set of permutations. We use the notation $S_n(R)$ to denote the set of all permutations in S_n which avoid every pattern in R.

Known Enumerative Results

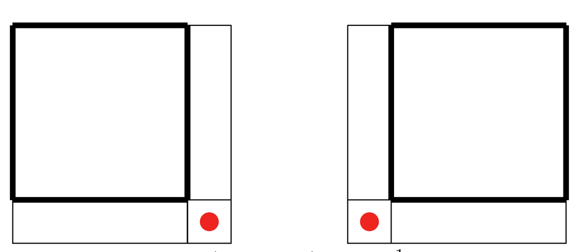
Here are some well-known formulae for counting permutations avoiding various sets of patterns:



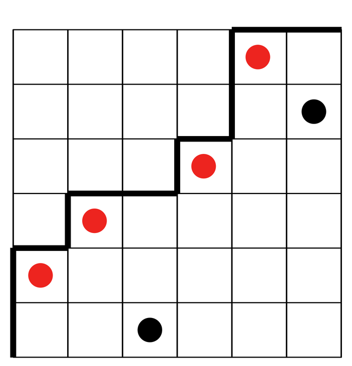
$S_n(12) = 1$



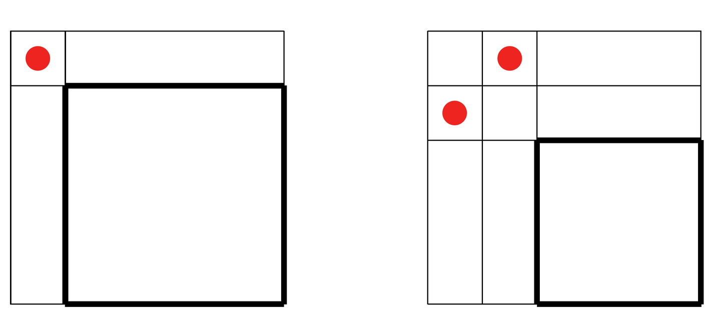
$S_n(132, 213, 231, 312) = 2$, $n \geq 2$



$S_n(213, 312) = 2^{n-1}$



$S_n(321) = S_n(213) = C_n = \frac{1}{n+1}\binom{2n}{n}$

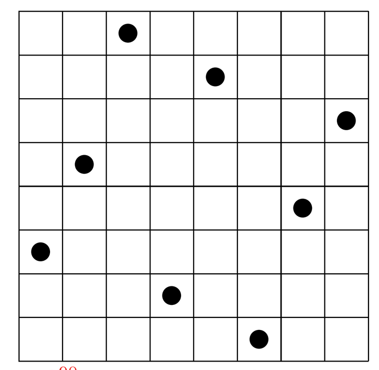


$S_n(123, 132, 213) = F_n$

where $F_0 = F_1 = 1$, $F_{n+2} = F_n + F_{n+1}$

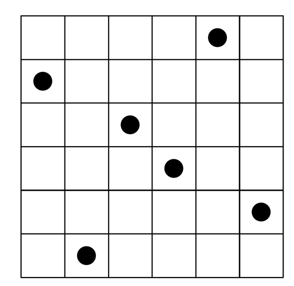
Symmetric Permutations

Next, we consider permutations whose diagrams are symmetric under subgroups of D_4, the dihedral group of order 8 (symmetries of a square). Here are the pertinent examples along with the notation we use:



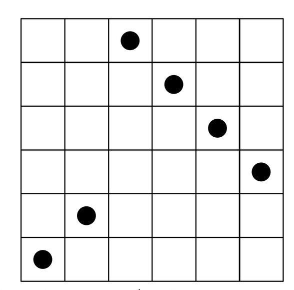
S_n^{90}: 90° rotational symmetry

$35827146 \in S_8^{90}$



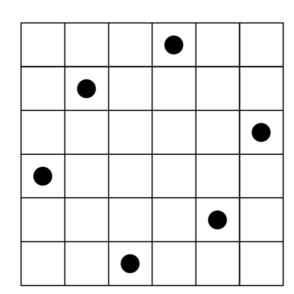
S_n^{rc}: 180° rotational symmetry

$514362 \in S_6^{rc}$



I_n: Involutions (reflection over $y = x$)

$126543 \in I_6$



I_n^{rc}: Involutions with 180° symmetry

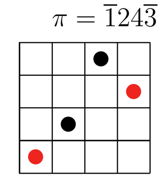
$351624 \in I_6^{rc}$

Counting S_n^{rc}

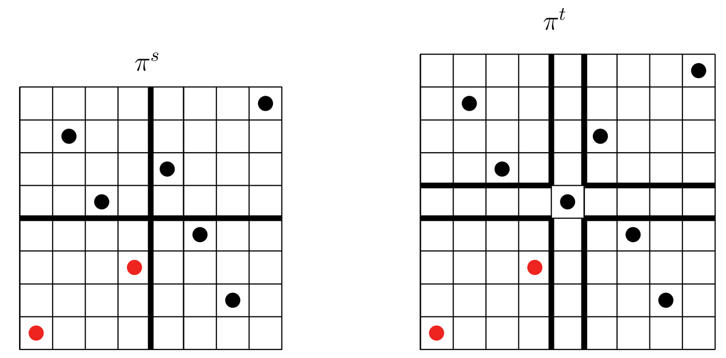
The set of signed permutations of length n, B_n, is the set of permutations of length n in which each entry may or may not have a bar above it. For example, $\overline{23}51\overline{4} \in B_5$. Clearly, $|B_n| = 2^n n!$.

We form bijections from B_n to S_{2n}^{rc} and to S_{2n+1}^{rc}:

$\pi = \overline{1}24\overline{3}$



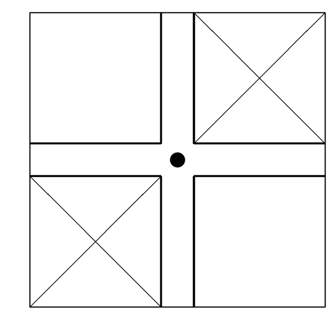
π^s π^t



Symmetric Pattern-Avoiding Permutations

Let R be a set of permutations. We use the notation $S_n^{rc}(R)$ to denote the set of all permutations in S_n^{rc} which avoid every pattern in R.

Example:



$$|S_{2n+1}^{rc}(123)| = C_n.$$

Theorem (Egge):

$$|S_{2n}^{rc}(123)| = \binom{2n}{n}$$

Summary of Results

There are $3! \times 4! = 120$ pairs of patterns (σ, τ), where $\sigma \in S_3$ and $\tau \in S_4$. Ignoring trivial cases and applying appropriate transformations to certain sets, this reduces to exactly twelve pairs for which the size of $S_n^{rc}(\sigma, \tau)$ is nontrivial to compute. We have combinatorial proofs for the sizes of all but one of these twelve sets, enumerated below.

Theorems (Lonoff, Ostroff):

σ	τ	$\lvert S_{2n}^{rc}(\sigma,\tau)\rvert$	$\lvert S_{2n+1}^{rc}(\sigma,\tau)\rvert$
123	2413	F_{2n+1}	F_{2n-1}
123	2431	$F_{n+3}+1$	$F_{n+2}-1$
123	3412	$2^{n+1}-(n+1)$	1
123	4231	$n2+1$	$\binom{n}{2}+1$
123	4312	6	1
123	1432	?	?
132	1234	T_n	T_n
132	2341	$F_{n+1}+1$	F_n+1
132	3412	$n+1$	$n+1$
132	4231	$n+1$	$n+1$
132	4321	$n+1$	$n+1$
132	3421	4	3

(Where $F_0 = F_1 = 1, F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$, and $T_0 = 1, T_1 = 2, T_2 = 3, T_n = T_{n-1} + T_{n-2} + T_{n-3}$ for $n \geq 3$.)

A New Fibonacci Identity

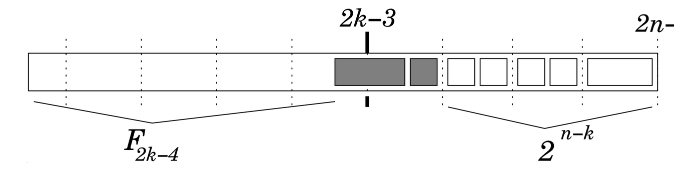
Our research also led us to a new identity involving sums of products of Fibonacci numbers.

$$F_{2n-2} + \sum_{k=1}^{n} F_{2k-4} 2^{n-k} = F_{2n}$$

This is equivalent to showing $\sum_{k=1}^{n} F_{2k-4} 2^{n-k} = F_{2n-1}$. How many ways can we tile a board of length $2n-1$ with dominoes and squares?

Answer 1: F_{2n-1}

Answer 2: We consider the odd fault lines, the vertical bars adjoining cells $2k+1$ and $2k+2$ on our board. Such a fault line is unbreakable if it is spanned by a domino, otherwise it is breakable. Where is the rightmost unbreakable odd fault line? How many ways can the spaces to the left be tiled? How many ways can the spaces to the right be tiled? The answer, illustrated in the diagram below, is the left side of the identity above.



This naturally generalizes to:

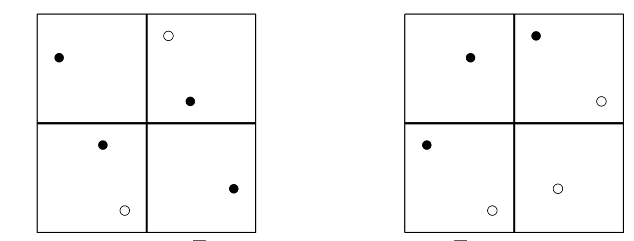
$$F_{mn+r} = F_r F_m^n + \sum_{k=1}^{n} F_{mk-m+r-1} F_{m-1} F_m^{n-k}$$

Model Proof

As an example of the combinatorial arguments used in our paper, we shall consider the specific question of $|S_n^{rc}(123, 4231)|$.

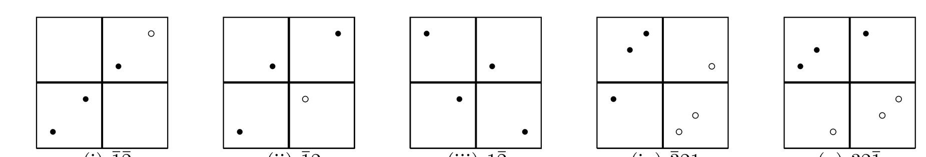
We determine whether some $\pi \in S_n^{rc}$ avoids 123 and 4231 by considering the corresponding signed permutation $\sigma \in B_m$ (for the appropriate value of m). With enough conditions, these σs are easy to count.

- The elements barred in σ form a consecutive set. That is, we never have $i < j < k$, with i and k barred in σ and j unbarred:



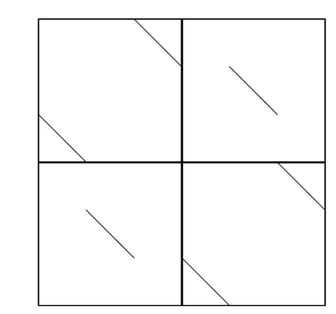
j precedes $\overline{k}$ $\quad$ $\overline{k}$ precedes j

- If π avoids 123 and 4231, then σ contains none of the following patterns:



(i) 12 (ii) 12 (iii) 12 (iv) 321 (v) 321

If σ has barred entries and meets these conditions, then σ is uniquely determined by what those barred entries are, yielding $\binom{n+1}{2}$ possibilities in the following form:



It is easy to see that, since 426153 avoids 123 and 4231, so do all permutations in the above form.

If σ has no barred entries, then π avoids 123 and 4231 if and only if σ avoids 321 and 132. By Simion and Schmidt, $|S_n(321, 132)| = \binom{n}{2} + 1$. Adding up both cases yields

Theorem (Lonoff, Ostroff):

$$|S_{2n}^{rc}(123, 4231)| = \binom{n+1}{2} + \binom{n}{2} + 1 = n2 + 1$$

Recall (as in $S_{2n}^{rc}(123)$) that if π is of odd length, then σ has no barred entries.

Theorem (Lonoff, Ostroff):

$$|S_{2n+1}^{rc}(123, 4231)| = \binom{n}{2} + 1$$

Acknowledgements

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