**Notes on Exponential growth and Decay Applications**

Several real world situations are modeled by the equation $y = Ce^{kt}$ where C is the original size of the population and k is the growth/decay constant. If $k > 0$, we have exponential growth. If $k < 0$, we have exponential decay.

## I) Compound Interest and Continuous Interest

To understand the continuous interest formula we first study the compound interest formula in great detail.

**The compound interest formula is** $B = P(1+\frac{r}{n})^{nt}$ **where *B* is the balance, *P* is the principle, *r* is the rate, *n* is the number of compoundings, and *t* is the time in years.**

### Explanation for the compound interest formula

The simple interest formula is $I = \Pr t$, where *I* is the simple interest, *P* is the principal, *r* is the rate, and *t* is time in years.

The balance in the account is $B = P + \Pr t = P(1+rt)$

Look at the following table

| Time in Years | Balance in the account |
|---|---|
| $t=1$ | $B_1 = P(1+r)$ |
| $t=2$ | $B_2 = B_1(1+r) = P(1+r)(1+r) = P(1+r)^2$ |
| $t=3$ | $B_3 = P(1+r)^3$ |
| $t=4$ | $B_4 = P(1+r)^4$ |
| | |
| | |
| $t=n$ | $B_n = P(1+r)^n$ |

To accommodate more frequent compounding of interest, we let *n* = number of compoundings per year and *t* be the number of years.

Then the rate per compounding is $\frac{r}{n}$.

The balance in the account after *t* years is $B = P(1+\frac{r}{n})^{nt}$ .

This is how we get the compound interest formula.

**The continuous interest formula is $B = Pe^{rt}$ where $B$ is the balance, $P$ is the principle, $r$ is the rate, and $t$ is the time in years.**

### Explanation for the continuous interest formula

In the compound interest formula, let $m = \frac{r}{n}$.

Then $B = P(1+\frac{1}{m})^{mrt} = P[(1+\frac{1}{m})^{m}]^{rt}$

Consider the quantity $(1+\frac{1}{m})^{m}$

| $m$ | 10 | 100 | 1000 | 10,000 |
|---|---|---|---|---|
| $(1+\frac{1}{m})^{m}$ | 2.59374 | 2.70481 | 2.71692 | 2.71815 |

Observe that $(1+\frac{1}{m})^{m} \to e$ as $n \to \infty$

So when the number of compoundings get so large as to approach infinity, the compound interest formula becomes $B = Pe^{rt}$. This is the famous continuous compound interest formula

**Compound interest problems**

**1)** A sum of \$9,000 is invested at an annual rate of 8.5%. Find the balance in the account after 3 years if the principal is compounded

a) annually
b) every six months
c) every month
d) continuously

**2)** An investment is made at an annual rate of 9.5% compounded quarterly. How long will it take for the investment to double in value?

**3)** Repeat the previous exercise using continuous compounding.

**4)** An investment of \$10,000 is compounded continuously. What is the annual rate that will produce a balance of \$25,000 in 10 years?

**5)** An investment is compounded continuously at an annual rate of 7.5%. Find the simple interest rate that would give the same balance at the end of 5 years.

**6)** When you buy a house for \$200,000 at 6% interest compounded continuously for 30 years, how much are you paying the bank over the life of the loan? What is your monthly payment?

**7)** A deposit is compounded continuously at an annual percentage rate of 7.5%. Find the effective yield. That is the simple interest that would yield the same balance at the end of the year. (Larson & Hostetler)

**II) Population Growth Problems**

In general, exponential growth and decay problems are modeled as $y = Ce^{kt}$ where $C$ is the original size of the population and $k$ is the growth/decay constant. If $k > 0$, we have exponential growth. If $k < 0$, we have exponential decay.

**8)** The number of fruit flies in a population after t hours is given by $y = 20e^{0.03t}$ where $t \geq 0$. How large is the population after 72 hours? (Larson & Hostetler)

**9)** A certain type of bacteria increases according to the model $y = 100e^{0.2197t}$ where $t \geq 0$ is time in hours. How large is the population after 10 hours? (Larson & Hostetler)

**10)** E-coli bacteria increases according to the model $y = 3000 2^{\frac{t}{20}}$ where $t \geq 0$ is time in minutes. How large is the population after one hour? (Beecher et al)

**11)** World demand for timber is increasing according to the model $y = 46.6(1.018)^t$ where $t$ is the number of years since 1981. What will the demand for timber be in 2015? (Beecher et al)

**12)** The population of a town increases according to the model $y = 100e^{0.2197t}$ where $t$ is time in years with $t = 0$ corresponding to 1980. Use the model to predict the population in 2020. (Larson & Hostetler)

**13)** The number of children under 18 in the US is decreasing according to the model $y = 35.37(0.99)^t$ where $t$ is time in years since 1964 and y is in millions. Use the model to predict the population in 2020. (Beecher et al)

**14)** The population of a town increases according to the model $y = 2500e^{kt}$ where $t$ is time in years with $t = 0$ corresponding to 1985. In 1935 the population was 3350.

a) Find the value of $k$
b) Predict the population in 2020. (Larson & Hostetler)

**15)** In 2001 the world population was 6.2 billion. Assume an exponential growth rate of 1.2% per year.

a) Write a function that models the population.
a) Estimate the population in 2020.
b) When will the population be 10 billion?
c) When will the population triple? (Beecher et al)

**16)** The population of California was 10,586,223 in 1950 and 23,668,562 in 1980. Assume the population grows exponentially.

a) Write a function that models the population.
b) Estimate the population in 2000 (and compare with actual population in 2000 to check accuracy of model). (Stewart et al)

## III) Radioactive Decay Model

Radioactive substances have a half-life. That is the amount of time it takes a certain amount of the material to become half the initial amount. For example, the half-life of radioactive Plutonium-239, which is used in bombs, is 25,000 years. So it takes 25,000 years for 4 grams of Pu-239 to become 2 grams.

**Problem**

**17)** Radioactive iodine has a half-life of 60 days. Many medical procedures require people to be exposed to radioactive iodine. Supposed the instrument malfunctions and exposes you to an initial amount C of radioactive iodine. Write an equation for the amount of iodine present at any time t after the accident. Use the equation to determine long will it take for the iodine to decay to a level of 20% of the original amount?

**18)** Radioactive plutonium ($Pu_{230}$) has a half-life of 24,360 years. If an accident releases 10 g of $Pu_{230}$ into the environment how much will remain after 1000 years?

**19)** Radioactive Strontium has a half-life of 28 years. How long will it take a 100 mg sample to decay to 30 mg.

**Carbon Dating**

When organic matter dies, its Carbon 12 content remains fixed while its Carbon 14 (radioactive carbon) content decays with a half-life of 5700 years. The ratio of Carbon 14 to Carbon 12 is 1 to $10^{12}$. That means $\frac{Carbon14}{Carbon12} = \frac{1}{10^{12}}$. To estimate the age of dead organic material the model used is $y = \frac{1}{10^{12}} 2^{\frac{-t}{5700}}$ where y is the ratio of Carbon 14 to Carbon 12 present in t years.

**20)** The $\frac{Carbon14}{Carbon12}$ ratio of a fossil is $\frac{1}{10^{13}}$. Find the age of the fossil. (Larson & Hostetler)

**21)** The statue of Zeus at Olympia, Greece is made of gold and ivory. The ivory was found to have lost 35% of its Carbon-14. How old is the statue? (Beecher et al)

**22)** The linen wrapping from one of the Dead Sea Scrolls had lost 22.3% of its Carbon-14 at the time it was found. How old was the linen wrapping? (Beecher et al)

## IV) Logistics Model

Some populations initially have rapid growth followed by a declining rate of growth. Virus and bacteria spread in this manner. One model for describing this type of growth pattern is the logistics function $y = \frac{M}{1 + (\frac{M}{c} - 1)e^{-kt}}$, where M is the maximum value and c is the initial size.

**23)** One student got a virus that is modeled by the function $y = \dfrac{5000}{1+4999e^{-0.8t}}$.

a) How many students will get infected in 5 days?
b) Classes will be canceled when at least 40% of the students get ill. After how many days will this happen? (Larson & Hostetler)

**24)** A lake is stocked with 500 fish and the fish population increases according to the logistics curve $y = \dfrac{10000}{1+10e^{-\frac{t}{5}}}$ where $t$ is measured in months.

a) How many fish are there after 30 days?
b) After how many months will the fish population be 2000? (Larson & Hostetler)

**V) Newton's Law of Cooling**

Suppose that a body with temperature $T_1$ is placed in surroundings with temperature $T_0$ different from $T_1$. The body will either cool or warm to temperature $T$ after time $t$ in minutes where $T = T_0 + (T_1 - T_0)e^{-kt}$.

**25)** An object in a room at 70°F cools from 350°F to 150°F in 45 minutes. Find the temperature after it has cooled for one hour. (Larson & Hostetler)
**26)** A thermometer is taken from a room at 72°F to the outdoors where the temperature is 20°F. The reading drops to 48°F after one minute. What is the reading after 5 minutes? (Larson & Hostetler)

**27)** A roasted turkey is taken from an oven when its temperature has reached 185°F and is placed on a table in a room where the temperature is 75°F. When will the turkey cool to the point we can eat it (say at 80°F). (Stewart et al)

**28)** Normal body temperature is 98.6°F. Immediately following death the body begins to cool. It has been determined experimentally that the constant in Newton's Law of cooling is approximately $k = 0.1947$ assuming time is measured in hours. Suppose that the temperature of the surroundings in 60°F. If a body is found with temperature 72°F, how long ago was the time of death? (Stewart et al)