

(Relatively) Painless Stress Tensors

Colin A. Shaw, University of Wisconsin, Eau Claire
shawca@uwec.edu

A graphical approach to introducing tensors

Adapted from Means (1976)

Context

Presentation: The activity is designed to be presented during class.

Audience: Undergraduate required structural geology course for majors. The activity is appropriate for students with a low-level mathematical background. Students should have taken at least trigonometry or pre-calculus and be familiar with vectors.

Skills and concepts that students must have mastered: Students should know the definitions of stress and strain and have a firm understanding of the strain ellipse (ellipsoid). They should have been introduced to the concept of force and understand that force is a vector quantity.

How the activity is situated in the course: This activity provides a bridge between the relatively concrete concept of the strain ellipse and the abstract concept of the stress tensor, represented (in two dimensions) by the stress ellipse. It is most appropriate for courses that introduce strain before stress.

Goals of the activity/assignment

Content/concepts goals for this activity: The exercise introduces the tensor operation that maps one vector onto another. In the case of stress the traction 'vector' is mapped onto a plane represented by the unit normal vector. The tensor representation of this operation is Cauchy's formula. The goal of the activity is to develop a concrete idea of what happens when a tensor operates on a vector in physical space. The activity segues into more abstract (and more practical) graphical constructions like the Mohr circle and lays the foundation for an intuitive understanding of tensor algebra.

Higher order thinking skills goals for this activity: Students should gain insight into the connection between vectors and tensor quantities in physical space and their abstract graphical and algebraic representations.

Description of the activity/assignment

* For a lucid description of a variant of the construction used here see Means (1976) pp. 53-58.

Materials:	<u>Instructor:</u>	<u>Students:</u>
	Transparency of figure 1	Transparency of figure 1
	Transparency of figure 2 (solution)	Straight edge
	Straight edge	Triangle or protractor
	Triangle or protractor	

1. Remind students to bring straight edge and triangle/protractor to class.
2. Distribute student handouts (figure 1)

Explain the construction.

3. The ellipse represents the state of stress at a point (center of the circle) with $\sigma_1 = 40$ MPa plunging $40 \rightarrow 090$ and $\sigma_3 = 20$ MPa plunging $50 \rightarrow 270$ σ_2 is N-S and horizontal (coming out of the page).
4. To help the students understand the stress ellipse I use the relatively concrete example of the strain ellipse that represents a deformed sphere. I introduce strain before stress, so they should know this, but I often remind the class by using a ball of silly putty or play dough to illustrate. The circle on the construction represents an isotropic state of stress (pressure) analogous to an undeformed sphere. I ask the class to lightly sketch the paths that points on the sphere would take as it was being squeezed to form the ellipse (parallel to the minor axis). "Easy, huh?"
5. Next I tell the class that the stress ellipse is more abstract. Instead of being a deformed sphere it represents the tails of arrows representing traction vectors - forces that act on specific planes. I remind them that for equilibrium there must be equal and opposite forces acting across every plane.

Do the construction

6. Have the students draw a plane with a N-S strike (perpendicular to the page). An example plane is specified on the handout (000 70). All constructions should be done carefully and accurately.
7. Have students draw the normal to the plane. Extend the normal at least to the isotropic circle.
8. To find the traction on the plane draw a construction line parallel to σ_3 from where the normal to the plane intersects the isotropic circle to the ellipse. Pause here and emphasize that this is the same path that a point on the 'undeformed' sphere would take during deformation.

9. Now the traction vector is drawn from the point where the construction line in (8) intersects the stress ellipse to the center of the circle (representing the point where stress is specified).
10. I use the analogy between the strain ellipse and stress ellipse to motivate a discussion about how both of the ellipses describe a transformation that relates two vectors. In the case of strain the two vectors are position vectors and in the case of stress they are a traction vector and a unit normal vector to a plane. This analogy should help students to form a more concrete picture of the abstract stress tensor.
11. Finally I have each student plot a different plane and find the stress on that plane. By comparing their results they see that (indeed) all the traction vectors radiate like spokes in a flattened wheel and their tails defined the stress ellipse.

Note: The discussion is very important in this exercise. If you just lead the students through the activity they will be baffled! The power of this activity is that it provides an opportunity to use the concrete strain ellipse as a model for the difficult abstraction of the stress tensor/ellipse. Depending on the (mathematical) level of students it may be possible to use the activity as a prelude to more rigorous treatment of tensor transformations.

Determining whether students have met the goals

Some time after working through this activity and after the Mohr circle has been introduced, I usually assign homework that requires students to draw planes and related tractions using correct angular relationships along with Mohr circles for several stress states. The homework is intended to build on the in-class activity and develop fluency in visualizing stress using several different representations. The homework is graded. The most common error in student work is drawing all traction vectors perpendicular to the corresponding stressed plane. In a typical class a small group of students find this activity very enlightening, a large group struggle to grasp its significance, and a small group never seem to comprehend what they did or why they did it.

Supporting references/URLs

Butler, R., Casey, M., Lloyd, G., McCaig, A., Teaching Structural Geology, Strain Ellipse

<http://earth.leeds.ac.uk/strain/strainellipse/strainellipse.html>

Hobbs, B.E., Means, W.D., and Williams, P.F., 1976, An Outline of Structural Geology, John Wiley & Sons, New York, pp. 12-17.

Means, W.D., 1976, Stress and Strain, Basic concepts of Continuum Mechanics for Geologists. Springer Verlag, Berlin, pp. 53-58.

Reish, N.E. and Girty, G.H, Visualizing Stress and Strain: Using Visual Basic 6.0 to Provide a Dynamic Learning Environment, San Diego State University Department of Geological Sciences,

<http://www.geology.sdsu.edu/visualstructure/vss/html/index.htm>.

Graphical Solution to Cauchy's Formula: Find the stress on a given plane

Stress Ellipsoid:
 σ_1 plunges 40 090
 $\sigma_1 = 40 \text{ MPa}$
 $\sigma_3 = 20 \text{ MPa}$

Example Planes:
N-S vertical
1. 000 70 E
2. 180 50 W
3. 000 40 E





