Moment Generating Function: used to calculate statistics for random variables (assume data sets are representative of statistical distribution of concentration) in solute transport.

1) Spatial Distribution – (plume)
   - Sample many locations for a given time; 1 time snapshot (where is mass?)
   - C varies in X, C(x)
   - Gives distribution of plume
   - Provides information on dimensions and location of solute
   - Lagrangian Approach- move along with solute; primarily 2-D and 3-D focus

2) Temporal Distribution – (BTC = breakthrough curve)
   - Sample many times at 1 location (when will mass arrive?)
   - C varies with time C(t)
   - Arrival and travel time (velocity and distance).
   - Provides information on arrival times of solute.
   - Eulerian Approach- watch solute pass by; primarily 1-D

Moment Analysis: evaluates spatial or temporal properties (statistics) about a plume.

1-D Spatial (absolute) M_n

(general) n^{th}: $M_n = \int cx^n dx = \Sigma cx^n \Delta x$

0^{th}: $M_0 = \int cdx = \Sigma c\Delta x$ (total mass)

1^{st}: $M_1 = \int cxdx = \Sigma cx\Delta x$ (mean location for the center of mass)

2^{nd}: $M_2 = \int cx^2 dx = \Sigma cx^2 \Delta x$ (spread of plume)

1-D Temporal (absolute) M_n

(general) n^{th}: $M_n = \int ct^n dt = \Sigma ct^n \Delta t$

0^{th}: $M_0 = \int cdt = \Sigma c\Delta t$ (total mass)

1^{st}: $M_1 = \int ctdt = \Sigma ct\Delta t$ (mean arrival time)

2^{nd}: $M_2 = \int ct^2 dt = \Sigma ct^2 \Delta t$ (degree of spreading)

Normalized (μ_n'): (always divide by the 0^{th} moment M_0); dx = spatial; dt=temporal

(general) $\mu_n' = M_n/M_0 = \int cx^n dx / \int cdx = \Sigma cx^n \Delta x / \Sigma c\Delta x$

Central (μ_n^c): (general); dx = spatial; dt=temporal

$\mu_n^c = \int (x-\mu_1')^n cdx / \int cdx = \Sigma (x-\mu_1')^n cdx / \Sigma c\Delta x$

Example of central (2^{nd} central or 2^{nd} central normalized)

μ_2^c = variance = $\sigma^2 = M_2/M_0 - \mu_1'^2 = \int(x-\mu_1')^2 c\,dx / \int c\,dx = \Sigma(x-\mu_1')^2 c\,dx/\Sigma c\Delta x$

$\mu_2^c = D2t$ or $d\mu_2/dt = 2D$; Where D is Dispersion Coefficient

Moment accuracy decreases with increasing order (3rd defines asymmetry or skewness).

Spatial Moment
M_0 = total mass (over distance)
Normalized M_1 = average location of plume center
M_2 = spread in distribution about that center
$d(\text{norm}M_1)/dt = dx/dt$ = velocity of plume

Temporal Moment
M_0 = total mass (through time)
Normalized M_1 = average time of arrival of plume (get velocity if you know the center of distribution)
M_2 = spread in time about that arrival time

Calculating Moments

You can use either rectangular or trapezoidal integration to perform moment analysis with a data set. Trapezoidal integration is usually seen as more accurate, and is given here. The formulas for the zeroeth and first normalized temporal moments are given. For spatial moments and moments of higher order, solve for yourself from these formulas and the notes above.

Given a data set with *n* number of concentration-time data points, beginning with point-'1':

$$\mu_0 = \sum_{i=2}^{n} \left(\frac{C_i + C_{i-1}}{2} \right) (t_i - t_{i-1})$$

$$\mu_1' = \frac{\sum_{i=2}^{n} \left(\frac{t_i C_i + t_{i-1} C_{i-1}}{2} \right) (t_i - t_{i-1})}{\mu_0}$$

Mean Arrival Time (MAT): = First normalized temporal moment

Mean Travel Time (MTT): = MAT – Avg. Injection Time of Tracer Pulse

$= \mu_1' - 0.5 t_o$

where t_o is the total injection time of tracer pulse. When solute mass balance is good (close to 100%) the zeroeth moment can be used as t_o.