Vectors and Vector Mathematics

Teacher Materials

This is the teachers' companion to two short articles for students. The first, "Math You Need—Vectors and Vector Addition," explains vectors as they are used in high school Physics—as an arrow. In the context of shifting GPS stations, the arrow points in the direction the station moves, while its length reflects its velocity. The article also shows how to add vectors visually. Students must be able to add vectors in order to analyze the GPS data used in "Velocity maps" and "Determining strain graphically."

The second article, "Two Faces of Vectors," connects students to more advanced math. It relates vectors-as-arrows to the matrices students learn in Algebra II, a subset of which are dubbed "vectors." This a) shows them a real, earthquake-related application of matrices; b) allows them to add vectors analytically; and c) builds towards understanding how the spreadsheet-based strain calculator operates. This second text also includes sections on adding and multiplying vectors as matrices. The simplest multiplication is by a simple number, a scalar. You might also, however, remember vector dot and cross products from college. Dot products are used in the strain calculator. Cross products are not, but are a logical extension of dot products—the two are often compared and contrasted. Both can be found by operating on matrices or by using sines and cosines. Resources are included to go even further so that mathematically adept students can explore the math that underpins the strain calculator.

If you are new to matrices, or haven't thought about them in a long while, consider delving into them for fun, and to support your students in advanced math. We did—and enjoyed it, thanks to resources like Khan Academy videos (free and online), the ever-useful Dummies Guide series (Dummies Guide to Linear Algebra), and summaries created by Vince Cronin for this unit on strain. We advise playing with this math; it's a lot like having a secret decoder ring. This article is organized to capitalize on a sequence of short Khan Academy videos.

Vectors — What Students Must Know to do the Strain Lessons

Adults use the word "vectors" in different contexts, from velocity vectors to ways by which diseases spread. Middle and high school students probably don't know what a vector is, period. Vectors are usually introduced in Physics. Students will need to add vectors to work through lessons in this unit on strain, but they can do it graphically, and resources explain what to do.

"Math you need—vectors and vector addition" explains vectors in a graphical or visual sense. The text also illustrates how to add vectors graphically by placing them head to tail. A resource for understanding vectors is the Khan Academy video "Introduction to Vectors and Scalars." (8:39. Search for "Khan vectors scalars.") The first 3:40 minutes of the Khan Academy video "Visualizing Vectors in 2 Dimensions" demonstrates adding vectors visually. (12:54. Search for "Khan visualizing vectors.")

Once students have added N – S and E – W velocity vectors, they must measure the new vector's direction—its azimuth--and its length. The new vector is the total horizontal velocity vector.

Questions or comments please education @ unavco.org   *Page 1*
Nancy West, Beth Pratt-Sitaula, and Shelley Olds, expanded from work initiated by Vince Cronin, Baylor University (Revisions David Thesenga)
Teacher version. Version of June 2022

Measuring Direction

If you give students a map showing north and a grid, as we describe in "Velocity maps" and "Determining strain graphically," they can use a protractor, a directional compass, or a compass or protractor app to measure the azimuth of the vector.

The task is simplest if you provide circular protractors. Students line up 0° on the protractor with north on the grid, and measure the vector's azimuth.

With a more common 180° semi-circular protractor, students might need help, depending on the general direction of the vector. Azimuths from 0° to 180° are straightforward. Beyond 180° (from due south, through west and back to north), they'll need to flip the protractor over and add 180° to their measurement.

Students can use a directional compass by lining their entire paper up with north, taping or holding it in place, and then lining the edge of the compass up with the vector. "Pure and simple graphing" for teachers shows how to do this. Two caveats: first, this will be more accurate with compasses that can be set to adjust for the magnetic declination, and second, desks with metal underpinnings can skew the results. But, this is another opportunity for students to practice with a compass.

Smartphone or tablet apps could serve the same purpose as protractors or directional compasses.

Measuring Length

A direct way to find the length of the vector is to mark its length on the edge of scrap of paper and to measure the scrap against the grid, where each block represents 1 mm.

Alternatively, students can wrestle with scaling. They can use a ruler to measure the vector's length and to measure the grid. (The grid will probably not print so that 1 mm on it is 1 mm in reality.) Then they'll set up a ratio to find the scaled measurement. For instance, if the vector measures 2.4 cm long (using a ruler) and the 10.0 mm mark on the grid is 3.0 cm away from the origin of the grid (using a ruler), students can find the vector's length this way:

Finding: mm of motion of a GPS station on the ground.

Known: Length of vector on paper: 2.4 cm; 3.0 cm on paper = 10.0 mm on the ground

$$\frac{x \text{ mm ground motion}}{2.4 \text{ cm vector on paper}} = \frac{10.0 \text{ mm ground motion}}{3.0 \text{ cm on paper}}$$

Cross multiply: $(x \ mm)(3.0 \ cm) = (2.4 \ cm)(10.0 \ mm)$

Isolate *x*:

$$x \ mm = \frac{(2.4cm)(10.0mm)}{3.0 \ cm} = 8.0 \ mm$$

Thus, a vector that measures 2.4 cm represents 8.0 mm of motion by a GPS station, given the scale of the grid in this example.

Students who have learned dimensional analysis might approach the problem this way:

$$\frac{2.4 \ cm \ vector \ on \ paper}{1} \ \text{X} \ \frac{10.0 \ mm \ on \ ground}{3.0 \ cm \ vector \ on \ paper} = 8.0 \ mm \ on \ ground$$

In summary, your students can determine the total horizontal velocity vector's direction and magnitude with a protractor and scrap of paper. But, they could do much more….

Vectors — Connecting to Mathematics

If your students can use one of the methods already described, they can work with GPS data and develop a sense of how much a region has moved and distorted over time. This sense can be qualitative *and* quantitative. This unit, however, can also use more sophisticated mathematics and show students how the math they study has utility. The rest of this article discusses other ways to work with vectors that bring in more sophisticated mathematics. It's intended to map out a route through resources that explain concepts; there is not enough detail here for this to stand alone. The student article, "Two faces of vectors" goes into the connections in more detail, and the resources referenced provide even more support.

Vectors — Finding Length with the Pythagorean Theorem

Students can calculate the length of the total horizontal velocity vector from N - S and E – W velocity vectors—if they have learned the Pythagorean Theorem. Most students learn it in 7th or 8th grade. They will find a worked example in "Velocity maps." The Khan Academy's video "Pythagorean Theorem" explains it and works a couple of examples. (10:46. Search for "Khan Pythagorean.") Students can check their results by searching online for "Pythagorean Theorem calculator."

Vectors, Sines and Cosines

With right triangles, vectors-as-arrows handily tie in sines and cosines, concepts some students learn at the end of Algebra 1 and all learn in Geometry. Remember that the sine is the side opposite the angle divided by the hypotenuse. The cosine is the side adjacent to the angle divided by the hypotenuse. The Khan Academy video "Basic Trigonometry" reviews the concepts. (9:17. Search for "Khan trigonometry.")

GPS data gives students the lengths of the opposite and the adjacent sides, and they determine the hypotenuse. The first two are the velocity vectors from time-series graphs, and the hypotenuse is the total horizontal velocity vector. Thus, they can find the azimuth of the total velocity vector either by measuring it (as described earlier) or by calculating the sine or cosine and using a calculator to find the angle. Why do this? First, students can get the same information in two different ways to check their results. Second, the process reinforces their math schoolwork. And, finally, computers use sine or cosine, not a protractor; they automate and speed up determining the azimuth of the vector. This is the kind of calculation that the spreadsheet-based strain calculator provides.
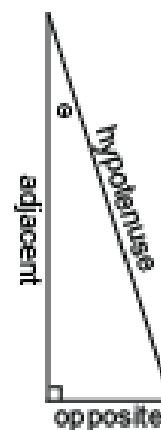
Adding Vectors Analytically

In the context of GPS-generated velocity vectors, you can think of the N - S and E - W velocity vectors as lying along the y-axis and x-axis respectively. With a clever way to describe the N – S and E – W vectors as units to be added analytically, you can combine them without drawing arrows. Unit vector notation provides that ability. The Khan

Academy video "Unit Vector Notation" shows these concepts with examples. (9:54. Search for "Khan unit notation.") Note that the video illustrates the reverse process, however: Mr. Khan decomposes a vector into its components parallel to the axes.

Vectors as (*x, y*) Ordered Pairs

Your students learned how to graph ordered pairs on x- and y- axes of a Cartesian coordinate system starting in about 4th grade. They will probably think of the pairs as a dot on a graph. However, a vector can also be represented by an ordered pair. The simplest case would be a vector that begins at the origin (0, 0) and ends at the ordered pair (*x, y*). However, vectors don't need to start at the origin; they could start at any pair of coordinates. Figure 1 shows the same vector with several different origins—but with the stipulated length and orientation. In these cases the head of the vector has the coordinates (2 + $x_{original}$, 3 + $y_{original}$).
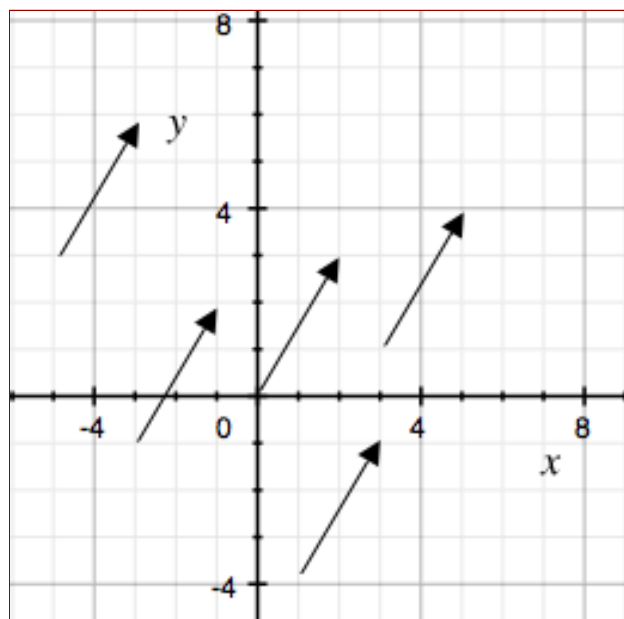

Figure 1. Identical vectors. The origin of one lies at the origin (0,0). Its head lies at (2, 3). The others have the same direction and magnitude but different points of origin.

Vectors as Matrices

The connection between vectors-as-arrows and vectors-as-matrices is not made in Khan Academy videos. The student guide "Two Faces of Vectors" walks through the connection—from ordered pairs marking the end of a vector to a sequence of numbers representing the ordered pair. The sequence can also be described as a column matrix—or a "vector" in Linear Algebra.

Your students begin learning to use matrices in Algebra II. GPS data allows them to see an application in science. That is, velocity vectors and bulkier matrices are used in the spreadsheet-based strain calculator of this strain module.

Operations on Matrices (and Vectors)

The rest of "Two faces of vectors" builds upon the mathematics of vectors using tables that sequence Khan Academy videos from Physics and Linear Algebra. Specific references go through dot and cross products, which you might remember from college Physics and your school's AP Physics students might encounter. For more advanced topics in Linear Algebra used in the strain calculator (e.g. transformations, transposes, inverses, determinants, Eigenvectors and Eigenvalues), the Khan Academy has two strands, "Matrix transformations" and "Alternative coordinate systems (bases)." *Linear Algebra for Dummies*, or a linear algebra textbook could be helpful resources.

Other resources are the articles "A summary of vectors and vector arithmetic" and "A summary of matrices and matrix math." The two provide examples to illustrate operations in three-dimensions.