Global Warming

A Zonal Energy Balance Model

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MATHEMATICAL FIELD

Calculus and Differential Equations.

APPLICATION FIELD

Climate Models and Environmental Science.

TARGET AUDIENCE

Students in an elementary mathematical modeling course or in a course of calculus with elementary differential equations, including numerical solutions for systems of ordinary differential equations.

ABSTRACT

Global warming is a response of the Earth to an imbalance between the energy absorbed and emitted by the planet. Earth is currently out of balance because increasing atmospheric greenhouse gases such as CO_2 reduce the planet's heat radiation to space, thus causing an energy imbalance. This imbalance causes Earth to warm in order to restore the energy balance. A Zonal Energy Balance Model (ZEBM) that tracks Earth's northern hemisphere temperatures is derived. Students working with this module will write a computer code to obtain numerical solutions of ZEBM, and will play the role of an environmental analyst by devising strategies for reduction of the growth of rate atmospheric CO_2 concentration in order to avoid "dangerous levels" of global warming.

PREREQUISITES

Precalculus and basic programing skills.

TECHNOLOGY

Access to a computer algebra system, such as Mathematica or MATLAB, is required for the main activities of this modulus.

AKNOWLEDGMENTS

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Introduction

The vulnerability of humanity to global temperature change has been well documented by responsible media and by scientific literature. The Intergovernmental Panel of Climate Change (IPCC), a scientific body appointed by the world's governments to advise them on the courses and effects of climate change, found that "continued emission of greenhouse gases will cause further warming and long-lasting changes in all components of the climate system, increasing the likelihood of severe, pervasive and irreversible impacts for people and ecosystems."[5]

Global warming is a response of the Earth to an imbalance between the energy absorbed and emitted by the planet. At a time of climate stability, Earth radiates as much energy to space as it absorb from sunlight. Earth is currently out of balance because increasing atmospheric greenhouse gases such as CO_2 reduce Earth's heat radiation to space, thus causing an energy imbalance. This imbalance causes Earth to warm in order to restore its energy balance. Measuring Earth's energy imbalance provides an invaluable tool for assessing global warming.

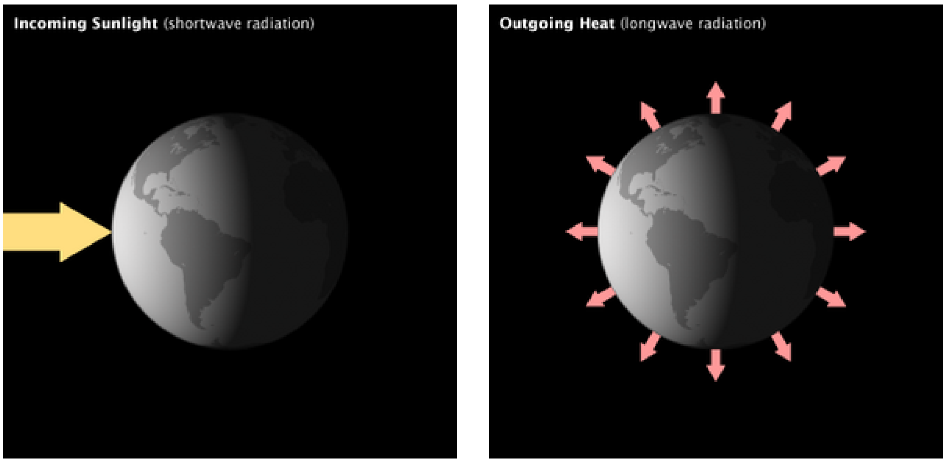

Figure 1: The energy that Earth receives from sunlight (shortwave radiation) is balanced by an equal amount of energy radiating into space (longwave radiation). Image from Pidwirny, M. (2013). Energy balance of Earth. Retrieved from http://www.eoearth.org/view/article/152458.

As controlled physical experiments with climate are difficult if not impossible to implement, mathematical modeling becomes an essential tool in the study of Earth's climate. The simplest models are the so-called Energy Balance Models (EBM) that estimate the changes in the climate system from an analysis of the energy budget of the

Earth.

This module introduces a Zonal Energy Balance Model (ZEBM) that describes the evolution of the latitudinal distribution of Earth's surface temperature. The model is used to estimate the levels of Earth's energy imbalance and corresponding zonal temperature increase. Students working on the assignments in the last section will write a computer code to obtain numerical solutions to ZEBM. They will play the role of an environmental analyst by devising strategies for reduction of the growth rate of atmospheric CO_2 concentration in order to avoid "dangerous levels" of global warming.

The module has been written with the intention of providing a brief introduction to the subject, directed mainly to instructors, to support the student's assignments. Instructors can use this content, together with the bibliographical references and other resources provided, to organize the presentation of the topics to their students during class time.

1 Derivation of the Model

We consider a Zonal Energy Balance Model that describes the evolution of the latitudinal distribution of Earth's surface temperature based on the assumption that the energy received by the planet from Sun's radiation must balance the radiation that the Earth is loosing to space. The model takes into account that part of the incoming solar energy is reflected back to space before entering the Earth, a phenomena know as *albedo*. It also incorporates the warming effect of the accumulation of CO_2 in the atmosphere.

We divide the Earth up into N latitudinal bands, assuming that the Earth is uniform with respect to longitude, with corresponding latitudinal mean surface temperatures T_j, $j = 1, \ldots N$ in °C, see Figure 2.

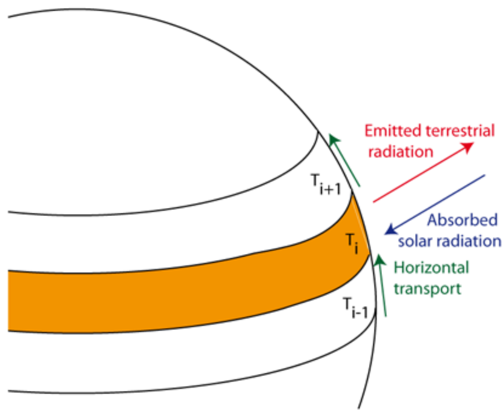

Figure 2: Representation of a Zonal Energy Balance Model for which the temperature T_i is averaged over a band of longitude. Image from Goosse H., P.Y. Barriat, W. Lefebvre, M.F. Loutre and V. Zunz, (2015). Introduction to climate dynamics and climate modeling. Online textbook available at http://www.climate.be/textbook

Our model includes three main components. Two of which are radiation mechanisms: the average amount of solar energy reaching a square meter of the Earth's surface in the $j-th$ band per year, and the average amount of heat energy emitted by a square meter of the Earth's surface in the $j-th$ band that reaches the stratosphere per year. They are denoted by E_{in}^j and E_{out}^j respectively.

Thirdly, we model the rate of change of heat energy between latitudes by a term of the form

$$K(\bar{T} - T_j),\ j = 1, \ldots, N, \tag{1}$$

where $\bar{T} := \frac{1}{N}\sum_{j=1}^{N} T_j$ is the mean global surface temperature. This is a transport mechanism satisfying the requirement that the direction of heat transfer is from a region of high temperature to another region of lower temperature. If the local temperature at a particular latitud is greater than the global mean, heat will be taken out of that latitude. Conversely, if the local temperature is colder than the global mean, that latitud will gain heat. Notice the similarity of (1) with Newton's law of cooling which states that the rate of heat loss of a body is proportional to the difference in temperatures between the body and its surroundings. The constant $K = 1.6B$ is a positive empirical constant (see [10, page 133]), where $B = 1.9\, Wyr/m^2\,°\mathrm{C}$, obtained by linear regression analysis of satellite data [3, Table 3], is the variation of the outgoing Earth's radiation per unit of change of global surface temperature.

Incoming Solar Radiation

The annual global mean incoming solar radiation or *insolation* is know as the *solar constant* Q, with current value $Q = 343 Wyr/m^2$. Part of the incoming solar energy is reflected back to to space before entering the Earth. This reflectivity is know as *albedo* and is modeled by a factor $\alpha \in (0, 1)$.

In order to describe the variations of the *albedo* and *insolation* due to the latitude, we represent the northern latitude by values y in the interval $[0, 1]$ given by the function $y = sin(\theta)$ with $0 \le \theta \le \pi/2$. Hence $y = 0$ corresponds to the equator, zero latitude, and $y = 1$ corresponds to the pole, latitude $\theta = \pi$.

Since the insolation Q is stronger at the equator than at the pole, an adjustment is introduced as a function $s(y)$ that represents the distribution of insolation over latitude. Although $s(y)$ is determined by astronomical calculations, it is uniformly approximated by $s(y) = 1 - .482(3y^2 - 1)/2$ for $0 \le y \le 1$ (see [7]). We will use the discrete version

$$s_j = s_N(j/N), j = 1, \ldots, N,$$

where $s_N(y) = 1 - .482(C_N y^2 - 1)/2$ and $C_N = 6N^2/(N + 1)(2N + 1)$.

The adjusted latitudinal insolation for the $j - th$ band is given by

$$Qs_j, \;\; j = 1, \ldots, N. \tag{2}$$

The constant C_N is determined by assuming that $\frac{1}{N}\sum_{j=1}^{N} s_j = 1$. This assumption is required to obtain that the average of the latitudinal insolation (2) equals the global mean insolation Q, that is $\frac{1}{N}\sum_{j=1}^{N} Qs_j = Q$.

Exercise 1. *Use the identity* $\sum_{j=1}^{N} j^2 = \frac{N(N+1)(2N+1)}{6}$ *to verify that* $\frac{1}{N}\sum_{j=1}^{N} s_j = 1$.

The *albedo* is also dependent of the latitude. While the *albedo* for open water is $\alpha_w = 0.32$, the *albedo* for snow-covered ice is $\alpha_s = 0.62$. We will assume that $\eta = 0.95$ is the location of the icecap, this correspond to 72^0 of latitude north, and consider that the albedo α_j is given by

$$\alpha_j = \begin{cases} \alpha_w, & \text{if } j < n \\ \frac{\alpha_w + \alpha_s}{2}, & \text{if } j = n \\ \alpha_s, & \text{if} j > n \end{cases},$$

where $n = [0.95N]$, and $[x]$ is the largest integer not greater than x.

Putting these assumptions together we obtain that the incoming solar energy in the $j - th$ latitudinal band is given by

$$E_{in}^j = (1 - \alpha_j) Q s_j, \;\; j = 1 \dots N. \tag{3}$$

Outgoing Heat Radiation

With cumulative CO_2 concentration and global mean surface temperature at the reference values c_0 and T_0 respectively, the outgoing Earth's radiation is given by

$$E_{out}^j = A_N + BT_j, \, j = 1, \dots, N. \tag{4}$$

Since the outgoing–longwave–radiation (OLR) can be measured by satellites, and there is a very good correlation between OLR and surface temperature, one can fit a straight line to the measured data to obtain B. This procedure gives $B = 1.9\, Wyr/m^2\,°\mathrm{C}$ (see [3, Table 3].) The constant A_N, given in units of Wyr/m^2, is obtained by assuming a stable climate system, i.e.

$$E_{in}^j - E_{out}^j + K(\bar{T} - T_j) = 0, \;\; j = 1, \dots, N, \tag{5}$$

for $\bar{T} = T_0$.

Exercise 2. *Show that*

$$A_N = (1 - \bar{\alpha}_N) Q - BT_0, \tag{6}$$

where $\bar{\alpha}_N = \frac{1}{N} \sum_{j=1}^{N} s_j \alpha_j$.

As the cumulative CO_2 concentration c exceeds the reference value c_0, the outgoing Earth's radiation is reduced by a term which, according to Arrhenius' greenhouse law

[1], is proportional to the logarithm of the concentration of infrared-absorbing gasses in the atmosphere[1]. We adopt the form

$$\tau B \log_2(c/c_0). \tag{7}$$

The parameter τ, given in units of °C, is the *equilibrium climate sensitivity* (ECS), defined as the equilibrium change in annual mean global surface temperature following a doubling of the atmospheric CO_2 concentration. Notice that when $c = 2c_0$ this term is reduced to τB, which is the reduction of Earth's radiation as a consequence of an increase of the global surface temperature of τ °C due to doubling of atmospheric CO_2 concentration. IPCC authors concluded that equilibrium climate sensitivity very likely is greater than 1.5 °C and likely to lie in the range 2 to 4.5 °C , with a most likely value of about 3 °C (see [8, 12]).

Putting together (4) and (7) we obtain the final expression for the outgoing Earth radiation in the $j-th$ latitudinal band:

$$E_{out}^j = A_N + BT_j - \tau B \log_2(c/c_0), \;\; j = 1, \ldots, N. \tag{8}$$

The Model

We now construct our model by stating that the rate of change of heat energy, given by $C\frac{dT_j}{dt}$, $j = 1, \ldots, N$, where C is the *effective heat capacity*, should be equal to that due to the incoming solar radiation minus that due to the outgoing Earth's radiation, plus the heat gained or lost from transport. Thus,

$$C\frac{dT_j}{dt} = (1-\alpha_j)Qs_j - (A_N + BT_j) + \tau B \log_2(c/c_0) + K(\bar{T} - T_j), \;\; j = 1, \ldots, N. \tag{9}$$

This is a spatially discrete version of the well known Budyko-Sellers zonal energy balance model (see [9]). Here $T_j(t)$, $j = 1, \ldots, N$ is the latitudinal mean surface temperature at time $t \geq 0$. The constant C is the *effective heat capacity*, the energy need to raise the global temperature by one degree Celsius. The heat capacity is commonly measured in watt-years per square meter per deg Celsius (Wyr/m^2 °C.) Its actual values depends on the medium under consideration and vary from $0.55\,Wyr/m^2$ °C for soil/atmosphere to

[1] In 1896, Swedish scientist Svante Arrhenius was the first to calculate the warming power of excess carbon dioxide (CO_2). From his calculations, Arrhenius predicted that if human activities increased CO_2 levels in the atmosphere, a warming trend would result. In its original form, Arrhenius' greenhouse law reads as follows: "if the quantity of carbonic acid [CO_2] increases in geometric progression, the augmentation of the temperature will increase nearly in arithmetic progression."

$90\,Wyr/m^2\,°\mathrm{C}$ for the ocean/atmosphere [6, page 15]. Since we are modeling the global climate system we will assume that the heat capacity is constant over the entire globe and equal to the weighted average $C = 64\,Wyr/m^2\,°\mathrm{C}$ obtained from the fact that the surface of the Earth is approximately 71% water.

We will solve system (9) subject to the initial condition

$$T_j(0) = \frac{(1-\alpha_j)Qs_j - A_N + T_0 K}{B+K}, \; j = 1, \dots, N. \tag{10}$$

Condition (10) is obtained by assuming a stable climate system, equation (5) with E^j_{in} and E^j_{out} given by (3) and (8) respectively, with $\bar{T} = T_0$ and $c = c_0$.

Global Warming

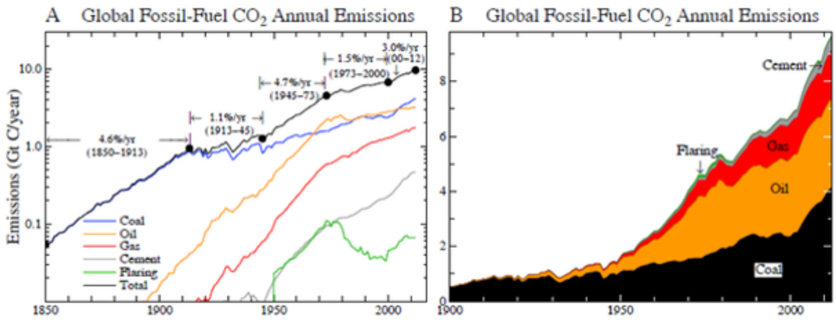

Figure 3: Global Fosil-Fuel Anual Emissions. Image from [4].

The United Nations Framework Convention on Climate Change [11] states its ultimate objective as: "stabilization of greenhouse gas concentrations in the atmosphere at a level that would prevent dangerous anthropogenic interference with the climate system." However, in reality the global emissions have accelerated (Fig. 3). The growth rate of fossil fuel emissions increased from 1.5%/year during $1980 - 2000$ to 3%/year in $2000 - 2012$.

The Framework Convention does not define a dangerous level for global warming. With the intention to provide a guideline for policy makers, authors of the Third Assessment Report of the IPCC presented the famous *burning embers diagram*, Figure 4, a graphic representation of the level of threat or risk associated with increases of global mean temperature. Classified into five *reasons for concern*, they include (i) risk to unique or threatened systems: the loss of endangered species, unique ecosystems, indigenous communities, and island nations, (ii) risk of extreme weather: more extreme heat waves, floods, and droughts, and more intense tropical cyclones, (iii) distribution of impacts:

the degree to which impacts are differentially harmful to different nations, regions, and populations, (iv) aggregate damages: a set of climate change impact metrics measuring economic costs, lives affected or lost, etc, (v) risk of large-scale discontinuities: "tipping point" phenomena, which could include the sudden loss or partial loss of the continental ice sheets, and abrupt changes in the modes of behavior of the ocean–atmosphere system impacting, e.g., water resource availability, among other possibilities. A probabilistic analysis [13] based in the *burning embers diagram* found a median *dangerous threshold* (DT) of 2.85 °C, with 90% confidence that the dangerous threshold was in the interval [1.45 °C, 4.65 °C].

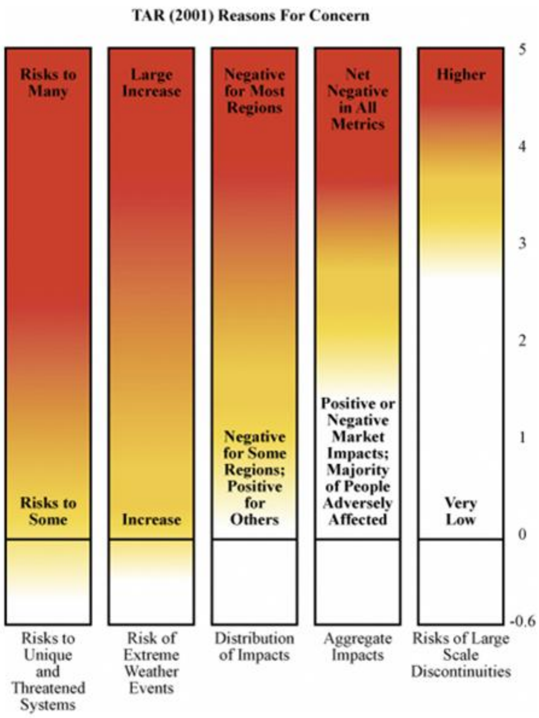

Figure 4: Five "reasons for concern" presented by the Third Assessment Report of the IPCC. Image from The Dictionary of the Climate Debate: http://www.odlt.org/dcd/.

The global annual mean concentration of CO_2 in the atmosphere increased by more than 40% from 280 parts per million (ppm) at the beginning of the Industrial Revolution to 369 ppm in 2000 [2]. The average growth rate from 1990 to 2000 was 0.41% per year. If the atmospheric CO_2 concentration continues to growth at this rate after the year 2000, the cumulative carbon emissions in the atmosphere will increase according to the exponential law $c = c_0(1.0041)^t$.

We replace this expression for c in (9) and solve (9)-(10) with $T_0 = 14\,°\mathrm{C}$ and $\tau = 1.5\,°\mathrm{C}$. The results are illustrated in Figures 5 & 6, showing an unlimited growth of the temperature distribution $T_j(t)$, $j = 1, \ldots, N$, and the average temperature $\bar{T}(t)$ respectively.

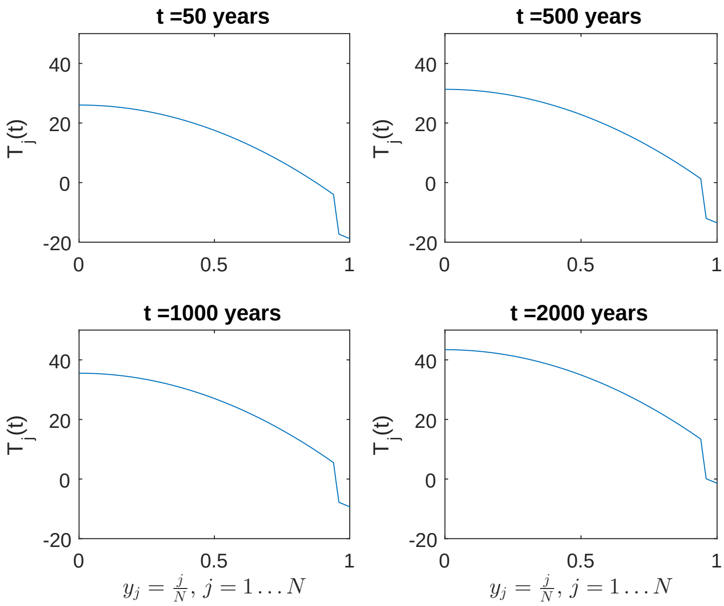

Figure 5: Temperature distribution $T_j(t)$, $j = 1, \ldots, N$, with $N = 100$ at different values for the number of years t from 2000, obtained by solving (9)-(10) using MATLAB's ODE solver *ode45* with $N = 100$, $T_0 = 14\,°\mathrm{C}$ and $\tau = 1.5\,°\mathrm{C}$.

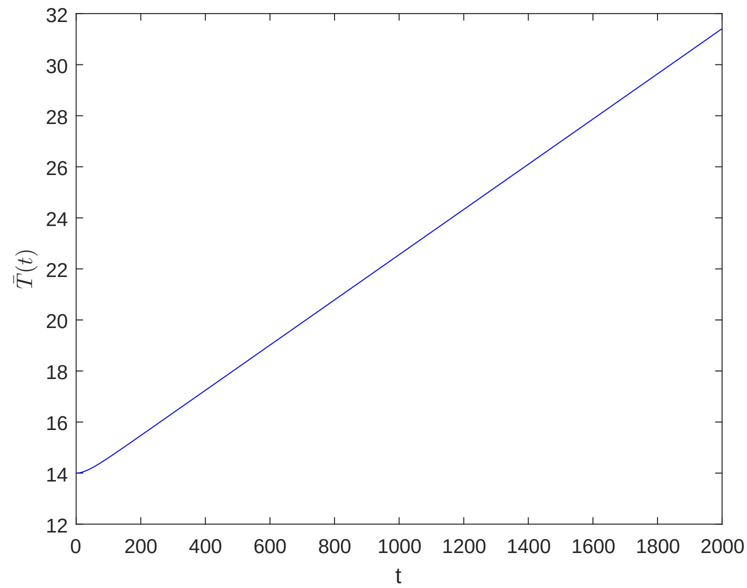

Figure 6: Global Mean Temperature $\bar{T}(t)$ obtained by solving (9)-(10) and taking the average $\frac{1}{N}\sum_{j=1}^{N} T_j(t)$ of the solution. We use MATLAB's ODE solver *ode45* with $N = 100$, $T_0 = 14\,°\mathrm{C}$ and $\tau = 1.5\,°\mathrm{C}$.

The only way that we could avoid surpassing any *dangerous threshold* assessment is to commit to a sustained reduction of the growth rate of atmospheric CO_2 concentration. We will show that this is possible by incorporating in our model a progressive reduction of the annual relative growth rate of commulative CO_2, given by the equation

$$\frac{dc/dt}{c} = re^{-\epsilon t} \tag{11}$$

which reduces the current rate r at $t = 0$ by the exponential factor $e^{-\epsilon t}$ for $t > 0$. The parameter $\epsilon > 0$ can be adjusted to avoid reaching any given estimate of dangerous threshold.

Exercise 3. *Equation* (11) *is a separable differential equation. Show that the solution is given by*

$$c = c_0 e^{\frac{r}{\epsilon}(1-e^{-\epsilon t})}. \tag{12}$$

To incorporate the CO_2 reduction strategy (11) to our model we introduce (12) into (9) to obtain

$$C\frac{dT_j}{dt} = (1-\alpha_j)Qs_j - (A_N + BT_j) + \frac{r\tau B}{\epsilon \ln 2}(1-e^{-\epsilon t}) + K(\bar{T} - T_j),\ \ j = 1, \ldots, N. \tag{13}$$

This is a linear system of non-autonomous (time-dependent forcing term) differential equations of the form

$$\frac{dT_j}{dt} + \beta T_j = \gamma_j + \delta(1 - e^{-\epsilon t}) + \lambda(\bar{T} - T_j), \ j = 1, \ldots, N, \tag{14}$$

with $\beta = B/C$, $\gamma_j = ((1-\alpha_i)Qs_j - A_N)/C$, $\delta = \frac{r\tau B}{\epsilon C \ln 2}$, and $\lambda = K/C$.

The following convergence theorem shows that it is possible to choose $\epsilon > 0$ such that global mean temperatures would remain below dangerous levels of global warming. The proof is left as an assignment for the interested reader.

Theorem 1. *Let $T_j(t)$, $j = 1, \ldots, N$, be a solution of* (14)-(10). *Then*

$$\lim_{t\to\infty} T_j(t) = T_j^*, \tag{15}$$

where

$$T_j^* = \frac{\gamma_j + \delta + \lambda \bar{T}^*}{\beta + \lambda}, \tag{16}$$

with,

$$\begin{aligned} \bar{T}^* &= \frac{\bar{\gamma} + \delta}{\beta} \\ \bar{\gamma} &= \frac{Q(1 - \bar{\alpha}_N) - A_N}{C} \\ \bar{\alpha}_N &= \frac{1}{N}\sum_{j=1}^{N} s_j \alpha_j. \end{aligned}$$

Moreover, $\bar{T}(t) < \bar{T}^*$ for $t \geq 0$.

Exercise 4. *Show that*

$$\bar{T}^* = T_0 + \frac{r\tau}{\epsilon \ln 2} \tag{17}$$

Theorem 1 and equation (17) shows that by choosing $\epsilon > 0$ such that

$$\frac{r\tau}{\epsilon \ln 2} = DT, \tag{18}$$

for a given *dangerous threshold* DT, the global temperature average $\bar{T}$ will remain below DT avoiding dangerous levels of global warming.

As an example lets assume we are in the year 2000 with reference values for the global mean surface temperature $T_0 = 14\,^\circ\mathrm{C}$ and cumulative carbon emissions $c_0 = 369\,ppm$. We consider the conservative scenario of an equilibrium climate sensitivity $\tau = 1.5\,^\circ\mathrm{C}$, dangerous threshold $DT = 2.85\,^\circ\mathrm{C}$, and initial growth rate $r = 0.41\%$ yearly.

Solving (14)-(10) with $N = 100$ and $\epsilon = r\tau/DT \ln 2 \approx 0.0031$ satisfying (18), we obtain Figure 7 describing the temperature distribution $T_j(t)$, $j = 1 \dots N$, at different values for the number of years t from 2000.

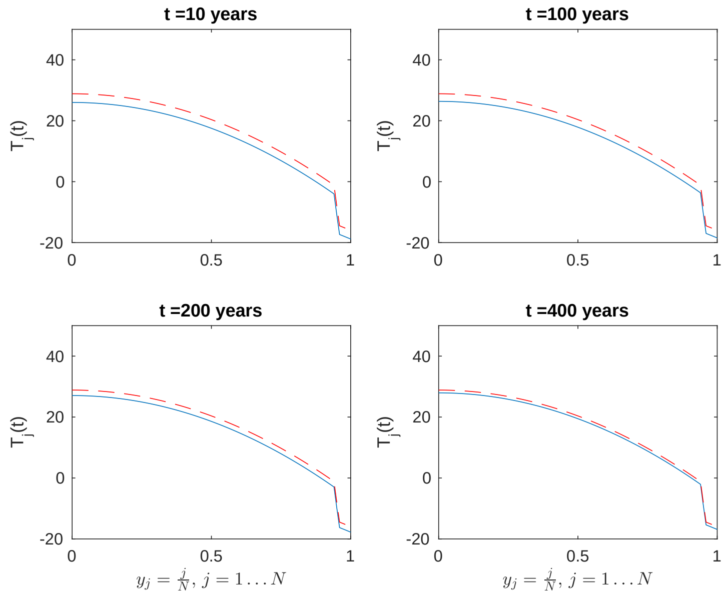

Figure 7: Temperature distribution $T_j(t)$, $j = 1, \dots, N$, with $N = 100$ at different values for number of years t from 2000, obtained by solving (14) with MATLAB's ODE solver *ode45*. We use the values $\tau = 1.5\,°\text{C}$, $DT = 2.85°\text{C}$, $r = 0.41\%$, and $\epsilon = r\tau/DT \ln 2 \approx 0.0031$ satisfying (18). The dashed line corresponds to T_j^* as given by (16).

Notice that Figure 7 suggests that $T_j(t) < T_j^*$, $j = 1, \dots, N$, for $t \geq 0$, where T_j^* is represented with a dashed line. This is not predicted by Theorem 1 which does predict $\bar{T}(t) < \bar{T}^*$ for $t \geq 0$ as verified by Figure 8.

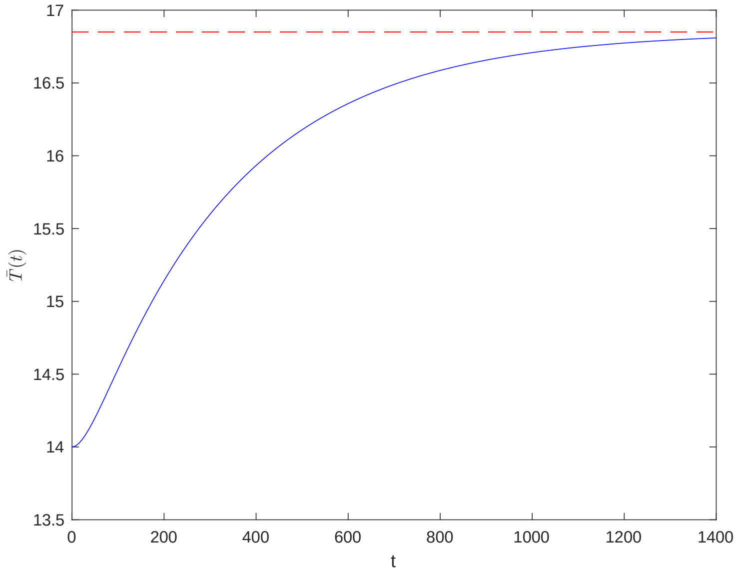

Figure 8: Global Mean Temperature $\bar{T}(t)$ obtained by solving (14) and taking the average $\frac{1}{N}\sum_{j=1}^{N} T_j(t)$ of the solution. The dashed line corresponds to the value of $\bar{T}^*$. We use MATLAB's ODE solver *ode45* with $N = 100$, $\tau = 1.5\,°\mathrm{C}$, $DT = 2.85°\mathrm{C}$, $r = 0.41\%$, and $\epsilon = r\tau/DT \ln 2 \approx 0.0031$ satisfying (18).

2 Assignments

Assignment 1. *Write a computer code to solve system* (14)-(10) *numerically. We recommend the MATLAB solver* **ode45***, or the Mathematica function* **NDSolve***, to obtain an accurate solution.*

Assignment 2. *Assume for simplicity that you are in the year 2000 with reference values for the global mean surface temperature* $T_0 = 14\,^\circ\mathrm{C}$*, cumulative carbon concentration* $c_0 = 369\,ppm$*, and initial growth rate of fossil fuel* $r = 0.41\%$*. Adopt your favorite values for the* equilibrium climate sensitivity τ *and* dangerous threshold DT*. With a choice of* $\epsilon > 0$ *satisfying* (18)*, use the script developed in Assignment 1 to verify that the global temperature average* $\bar{T}(t)$ *will remain below* $T_0 + DT$ *for* $t > 0$*.Describe your results with graphical outputs similar to Figures 7 & 8.*

References

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